CS 170
Efficient Algorithms and Intractable Problems

Lecture 2:
Divide and Conquer I, Asymptotics

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The classroom annotation were not saved on the notes.

These slides instead include typed out solutions, highlighted by yellow-colored background.

Announcements

- Office hours starts Tuesday (after class). Check the calendar.

- Discussions start this week, first one is on Wednesday
  - Leetcode discussion section. Check the calendar

- Homework 1 is out, Due on Saturday.
  - Homeworks are graded for feedback, not credit.

- Class quizzes will be drawn from homeworks and discussions. So do the homework!

Recap of last time

Introductions all around!

Our motivating questions about algorithms:

- Does it work?
- Is it fast?
- Can I do better?

Technical content:

- Arithmetic and Big Oh notation
- Intro to Divide and Conquer
- First attempt at fast multiplication

→Still didn't beat $O(n^2)$

- We saw Karatsuba's trick

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- Intro to Divide an
- First attempt at fas

→ Still didn't beat O

(simplify: assume even n)

The algorithm

Break up the multiplication of two integers with n digits into multiplication of integers with $n/2$ digits:

$$[x_1 x_2 \cdots x_n] = \overbrace{[x_1, x_2, \cdots, x_{n/2}]}^{a} \times 10^{\frac{n}{2}} + \overbrace{[x_{n/2+1} x_{n/2+2} \cdots x_n]}^{b}$$

$$\begin{aligned} x \times y &= \left(a \times 10^{\frac{n}{2}} + b\right)\left(c \times 10^{\frac{n}{2}} + d\right) \\ &= \underbrace{(a \times c)}_{\text{P1}} 10^{n} + (\underbrace{a \times d}_{\text{P2}} + \underbrace{c \times b}_{\text{P3}}) 10^{n/2} + \underbrace{(b \times d)}_{\text{P4}} \end{aligned}$$

One n-digit multiplication → Four $n/2$-digit multiplications

Recap of last time

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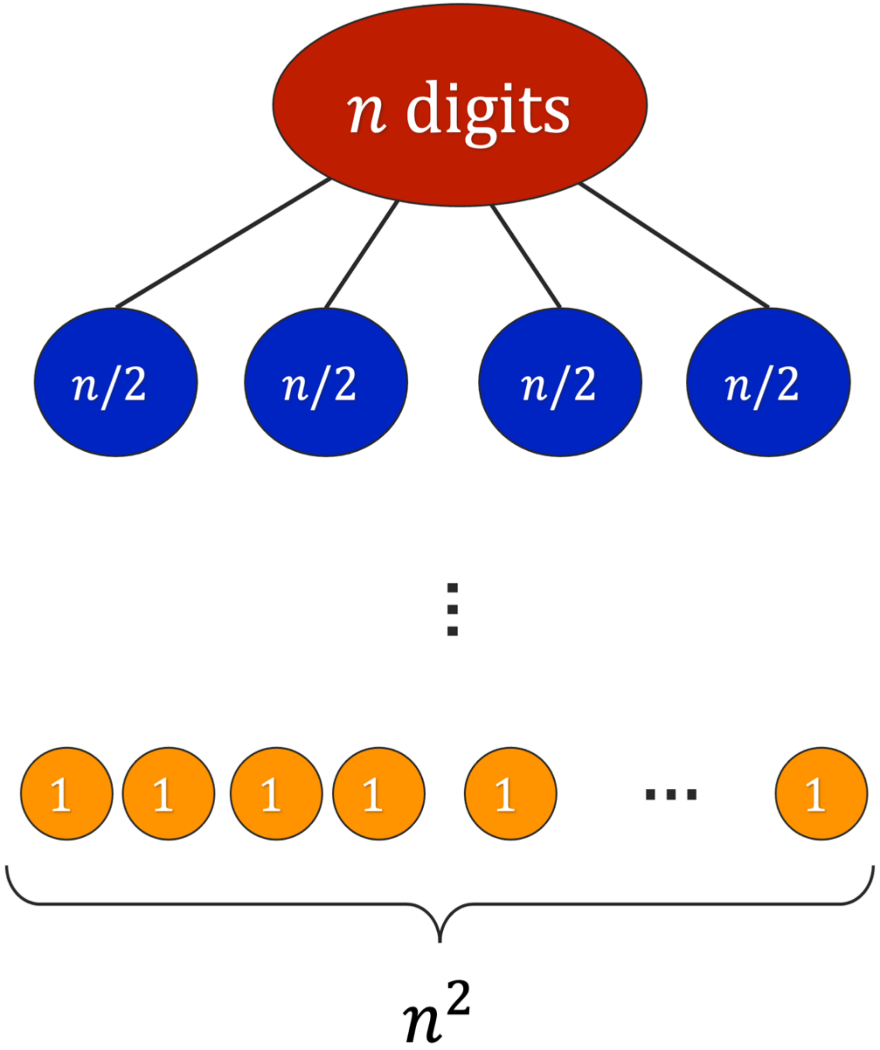


Layer	# of digits	# problems
0	n	1
1	$n/2$	4
⋮	⋮	⋮
t	$\frac{n}{2^t}$	4^t
⋮	⋮	⋮
$\log_2(n)$	1	$4^{\log_2 n} = n^2$

This lecture

- More details on Karatsuba's algorithm with $O(n^{1.6})$

→Using divide and conquer, but this time better!

- Reviewing $O(\cdot)$ and $\Omega(\cdot)$ notation formally.

- Recurrence relations and a useful theorem for solving them!

Karatsuba's Idea

Divide and Conquer indeed can lead to a faster algorithm!

$$
\begin{aligned}
x \times y &= \left(a \times 10^{\frac{n}{2}} + b\right)\left(c \times 10^{\frac{n}{2}} + d\right) \\
&= \underbrace{(a \times c)}_{\text{P1}} 10^{n} + (\underbrace{a \times d}_{\text{P2}} + \underbrace{c \times b}_{\text{P3}}) 10^{n/2} + \underbrace{(b \times d)}_{\text{P4}}
\end{aligned}
$$

The issue is that we are creating 4 sub-problems.
What if we could create fewer subproblems?

Main idea: Could we write P2+P3 using what we compute in P1 and P4, and at most one other $n/2$-digit multiplication?

Well... Gauss had used this trick too, for complex numbers...

Karatsuba's Clever Trick

Let us only compute 3 multiplications with $n/2$ digit numbers:

- Q1: $a \times c$
- Q2: $b \times d$
- Q3: $(a + b)(c + d)$

Expressing P2+P3 differently

$$ad + cb = (a + b)(c + d) - \mathrm{ac} - \mathrm{bd}$$

Discuss: Write down the decomposition to subproblems

Write as only 3 multiplications of $n/2$ digit number:

$$x \times y = \left(a \times 10^{\frac{n}{2}} + b\right)\left(c \times 10^{\frac{n}{2}} + d\right)$$

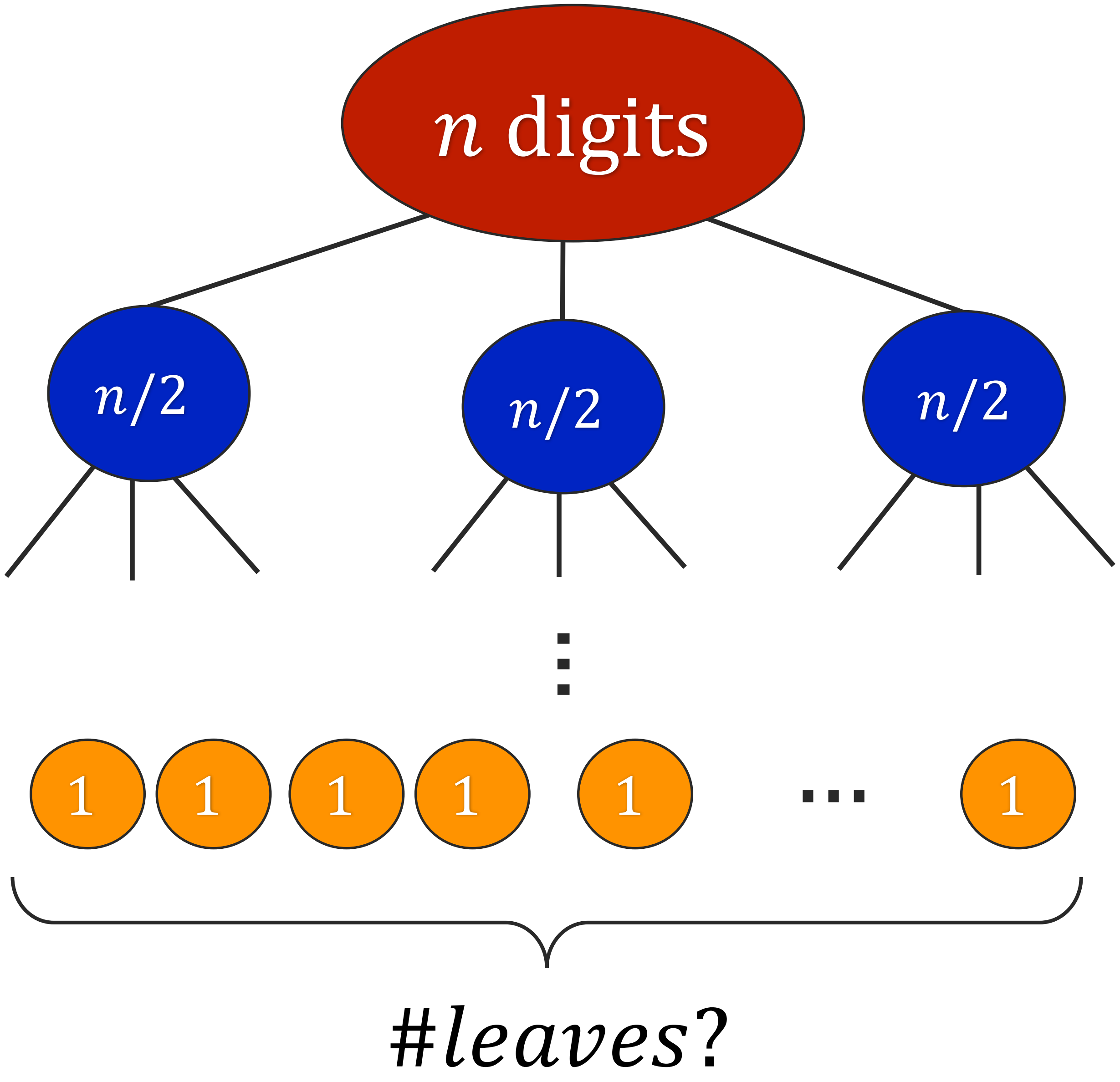
$$= \underbrace{(a \times c)}_{\text{Q1}}10^{n} + \underbrace{(a \times d + c \times b)}_{\text{Q3}-\text{Q1}-\text{Q2}}10^{n/2} + \underbrace{(b \times d)}_{\text{Q2}}$$

What is the runtime of Karatsuba's algorithm?

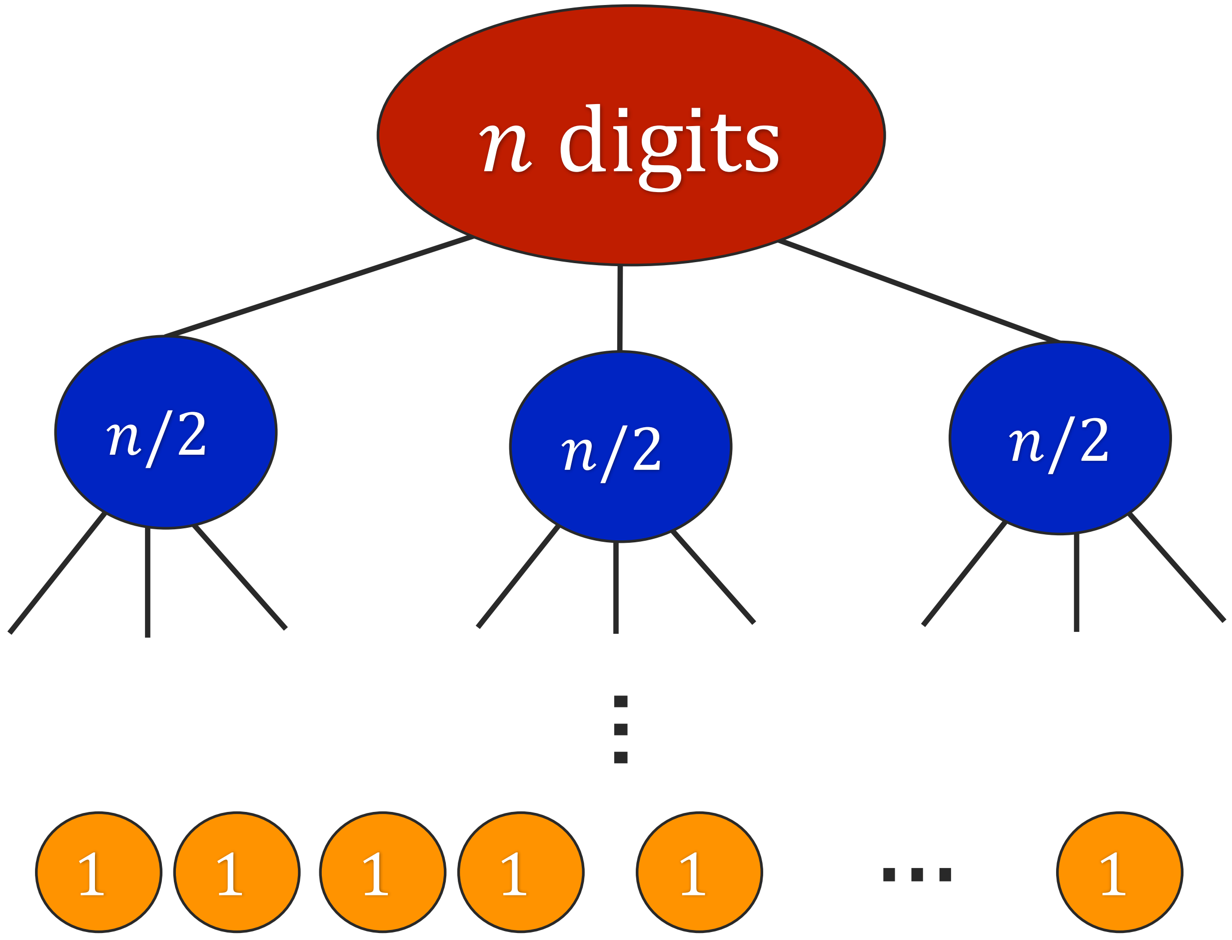
Less formally, how many 1-digit multiplications do we do in Karatsuba's algorithm?

Same approach as last lecture,

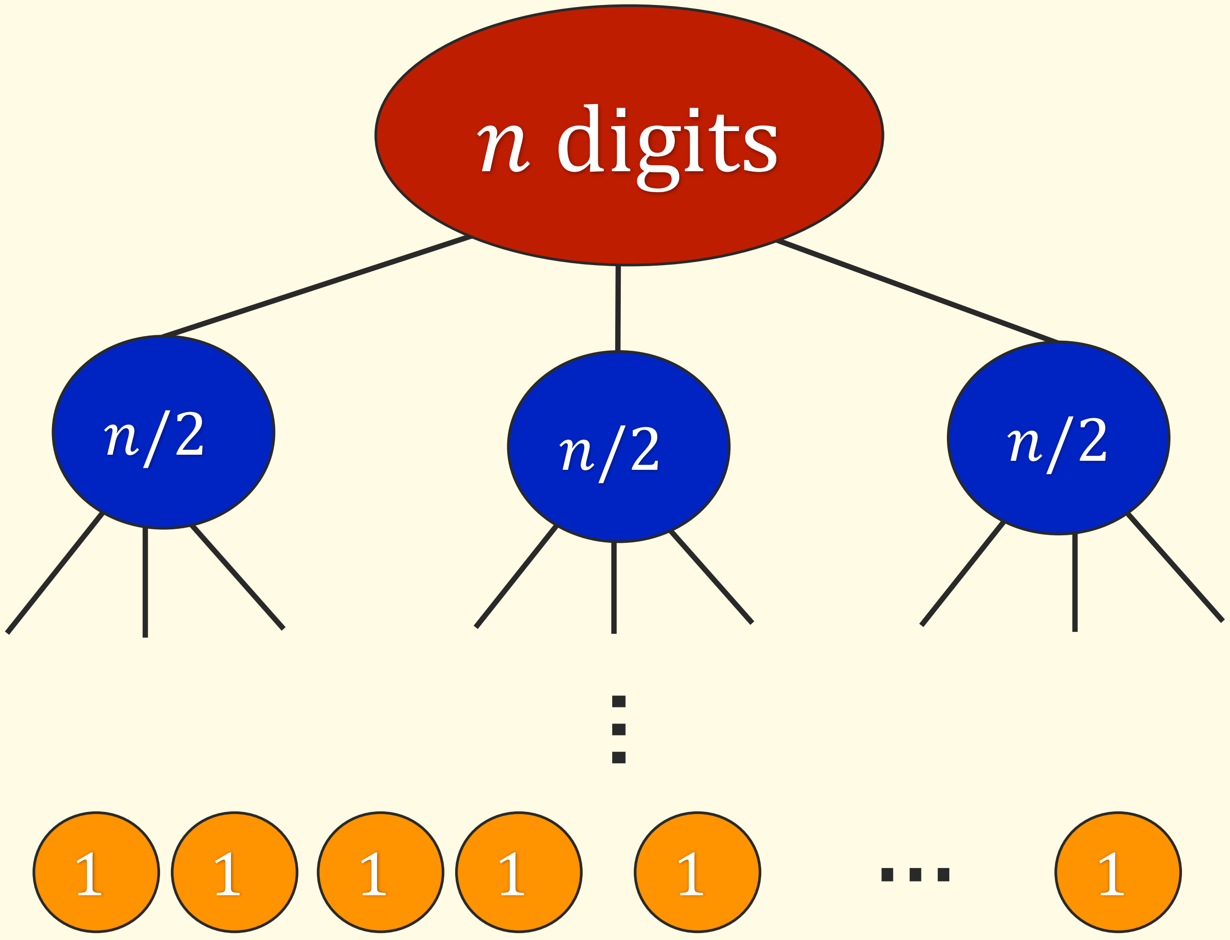
this time our branching factor is 3 instead of 4



Layer	# of digits	# problems
0	n	1
1	$n/2$	3
⋮	⋮	⋮
t	#digits per node	# nodes
⋮	⋮	⋮
Depth	1	# leaves



Layer	# of digits	# problems
0	n	1
1	$n/2$	3
$\vdots$	$\vdots$	$\vdots$
t		
$\vdots$	$\vdots$	$\vdots$
	1	



$$3^{\log_2 n} = n^{\log_2(3)} \approx n^{1.6}$$

Layer	# of digits	# problems
0	n	1
1	$n/2$	3
⋮	⋮	⋮
t	$n/2^t$	3^t
⋮	⋮	⋮
$\log_2 n$	1	$3^{\log_2 n} = n^{\log_2(3)}$

Other Algorithms

- **Karatsuba** (1960): $O(n^{1.6})$! Saw this!
- **Toom-3/Toom-Cook** (1963): $O(n^{1.465})$

(advanced)

Divide and conquer too! Instead of breaking into three n/2-sized problems, break into five n/3-sized problems.

Hint: Start with 9 subproblems and reduce it to 5 subproblems.

- **Schönhage–Strassen** (1971):
  - Runs in time $O(n \log(n) \log \log(n))$
- **Furer** (2007)
  - Runs in time $n \log(n) \cdot 2^{O(\log^*(n))}$
- **Harvey and van der Hoeven** (2019)
  - Runs in time $O(n \log(n))$

What about binary representation?

We used base 10 so far

→Counted the # of 1-digit operations, assuming adding/multiplying single digits is easy (memorized our multiplication table!)

What if we use base 2?

→We would want to count # of 1-bit operations.

How do we alter Karatsuba's algorithm for binary numbers?

N-bit integer multiplications

Easy to compute 10^k in base 10. In base 2, it is easy to compute 2^k.

$$[b_1 b_2 \cdots b_n] = \left[b_1, b_2, \cdots, b_{n/2}\right] \times 2^{n/2} + [b_{n/2+1} b_{n/2+2} \cdots b_n]$$

$$\boxed{\underbrace{b_1 \quad b_2 \quad b_3 \quad \cdots}_{b^L} \,\Big|\, \underbrace{\cdots b_{n-2} \quad b_{n-1} \quad b_n}_{b^R}}$$

$$a \times b = (a^L \times b^L)2^n + (a^L \times b^R + a^R \times b^L)2^{n/2} + (a^R \times b^R)$$

$$= \cdots$$

Practice: Complete this equation the Karatsuba's way and rederived $O(n^{1.6})$ runtime for multiplying two n-bit numbers. Might see this on HW or Discussion!

Details we skipped

Technically

- We only counted the number of 1-digit problems
- There are other things we do: adding, subtracting, …
- Shouldn't we account for all of that?

Absolutely!

- We should be more formal, and we will be next!
- In this case, additions/subtractions end up in lower order terms
- Don't affect $O(.)$.

Asymptotic Notations
More Formally

Runtime of Algorithms Asymptotically

Suppose an algorithm with input size n takes

$$T(n) = 5n^2 + 20n\log(n) + 7 \quad \text{ms}$$

$T(n) \in O(n^2)$ also commonly written as $T(n) = O(n^2)$

Why is it a good idea to just say this is $O(n^2)$?

- Constants like 5, 20, 7, depend on the platform and computer.
- Makes it easier to compare the performance of algorithms on large inputs
- Makes algorithm analysis easier
- Sometime clever tricks and representations improve the constants anyway.

Definition of O(…)

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- We say “$T(n)$ is $O\left(g(n)\right)$” if and only if

  for large enough n,

  $T(n)$ is *at most* some constant multiple of $g(n)$.

Definition of O(…)

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- We say "$T(n)$ is $O\big(g(n)\big)$" if and only if

$$\boxed{\begin{array}{c}\text{There exists } c \text{ and } n_0 > 0 \\ \text{Such that for all } n \geq n_0, T(n) \leq c \cdot g(n)\end{array}}$$

- Always give the tightest and simplest O( ) you can.

→ e.g., $5n^2 \in O(n^3)$ and $5n^2 \in O(2n^2 + n)$ too,

→ but give the best bound of $O(n^2)$.

Example

Prove that for $\mathrm{T}(n) = 2n^2 + 2$, we $\mathrm{T}(n) \in O(n^2)$

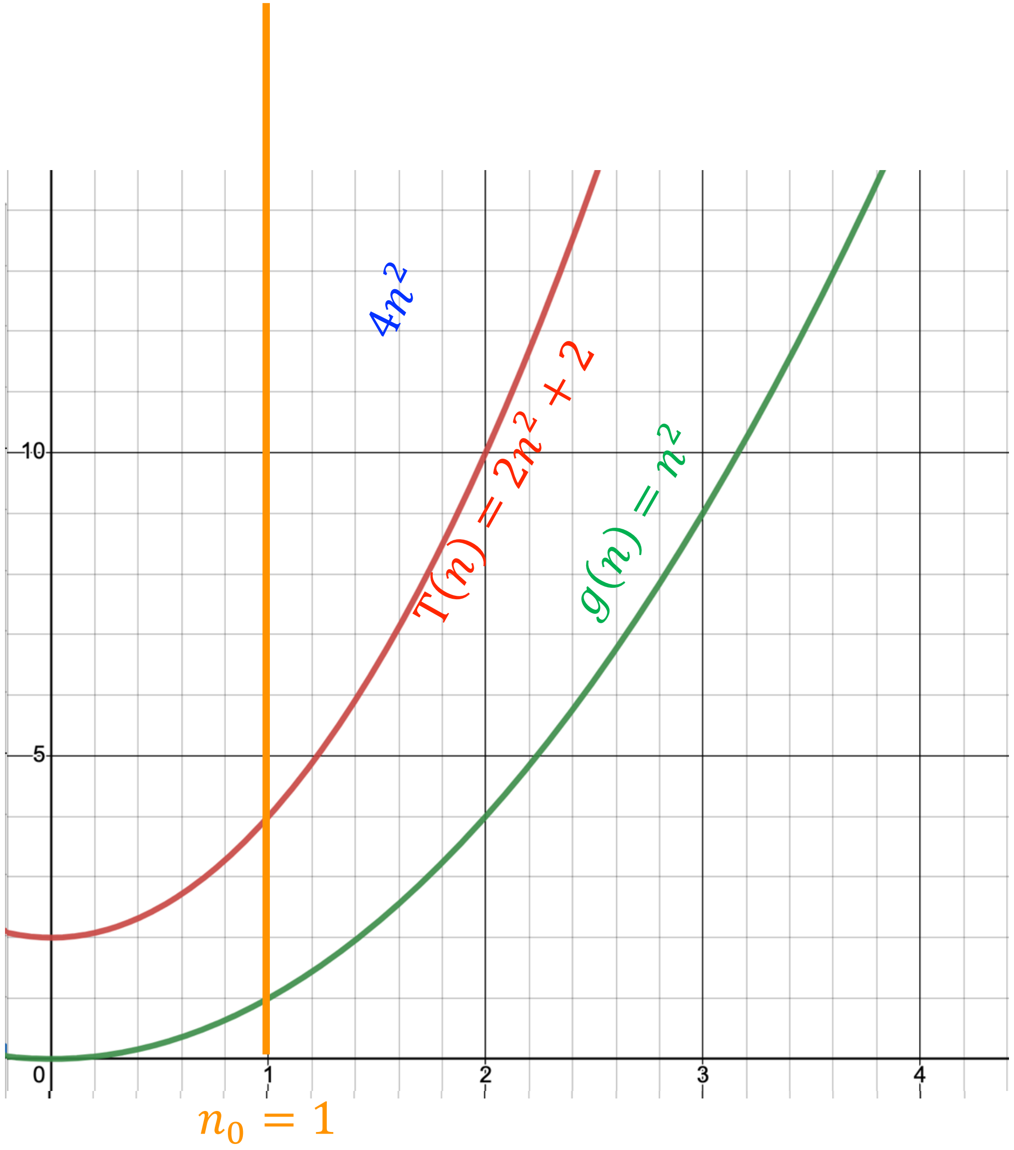
Even though $\mathrm{T}(n)$ is larger than n^2 always, we can find $c = 4$ and $n_0 = 1$, such that all $n > n_0$

$$2n^2 + 2 \leq 4n^2$$

How do you prove the above inequality?

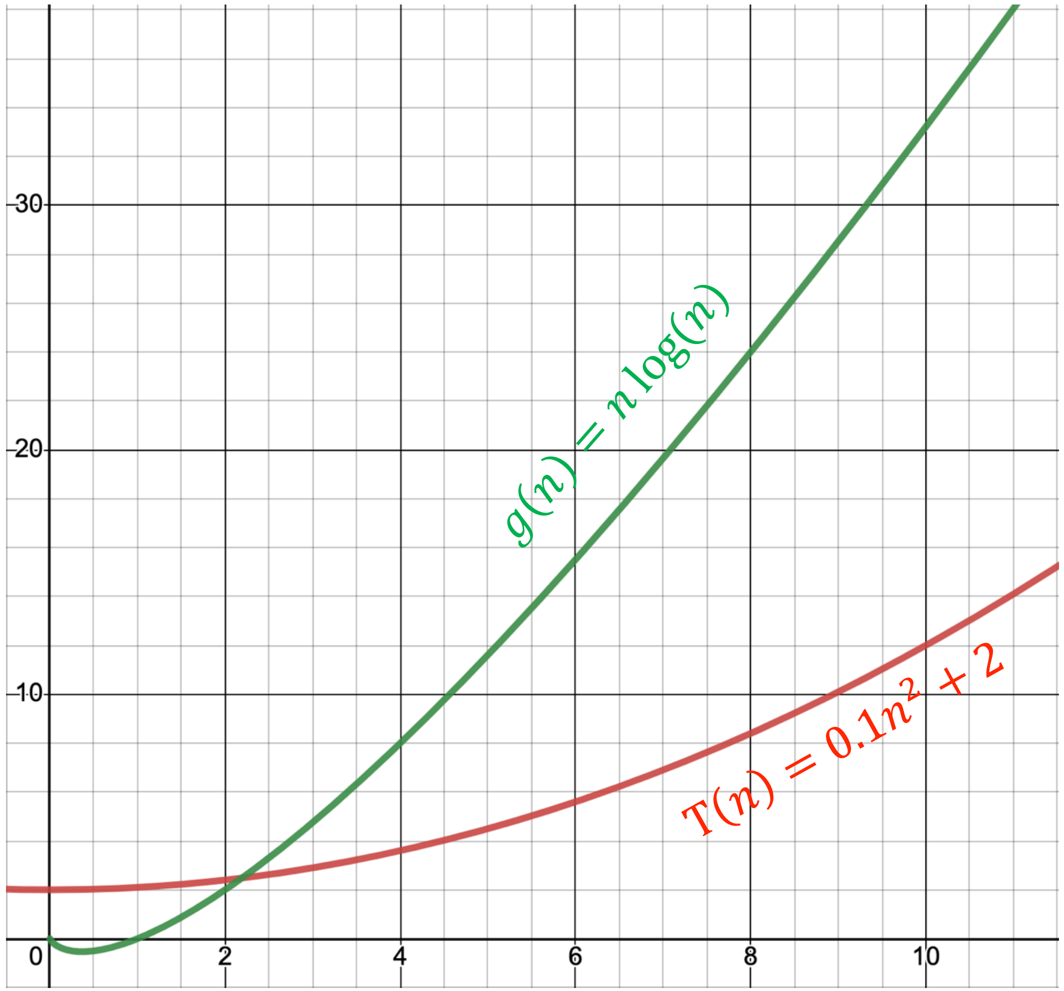
Whatever (correct!) math you like!

- E.g., equal at n_0 and RHS has larger derivative.



BEWARE! of pictures!

The picture seems to imply that for $\mathrm{T}(n) = 0.1n^2 + 2$ we have that $\mathrm{T}(n) \in O(n\log(n))$!



What's wrong with this argument and relying on pictures?

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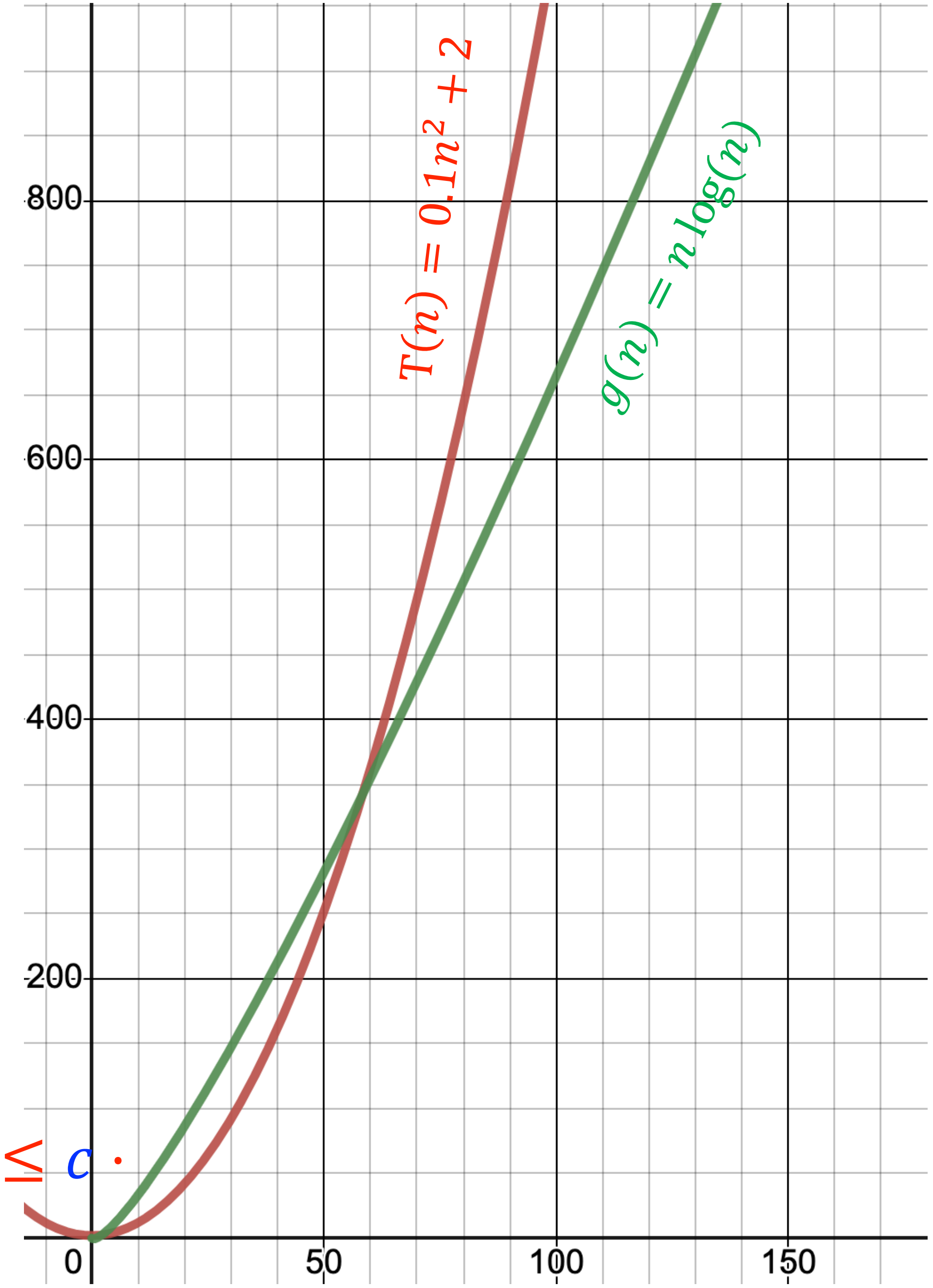


What's wrong with this argument and relying on pictures?

That is why you should come up with c and n_0 and mathematically prove that for all $n \geq n_0, T(n) \leq c \cdot g(n)$.

How to prove $0.1n^2 \notin O(n)$!

- Proof by contradiction:
- Suppose that $0.1\, n^2 \in O(n)$.
- Then there is <u>some positive c and n_0</u> so that:

$$\forall n \geq n_0, \qquad 0.1n^2 \leq c\, n$$

- Divide both sides by n:

$$\forall n \geq n_0, \qquad 0.1n \ \leq c$$

- That's not correct. Let $n = n_0 + 10\text{ c}$
  - Then $n \geq n_0$, but $0.1n \ > c$.
- Contradiction!

Recap of Proof Techniques

To prove $T(n) \in O(g(n))$:

- You have to come up with c and n_0 so that the definition is satisfied.

To prove $T(n) \notin O(g(n))$

- You have to rule out **all possible** c and n_0.
- One approach is to use **proof by contradiction**:
  - Suppose there exists a c and an n_0 so that the definition ***is*** satisfied.
  - Derive a contradiction,

  → e.g., by finding large enough n (as a function of c and n_0), for which the definition is not satisfied.

$\Omega(\ldots)$ means lower bound

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- We say "$T(n) \in \Omega(g(n))$" if and only if

$$\text{There exists } c \text{ and } n_0 > 0$$

$$\text{Such that for all } n \geq n_0,\ c \cdot g(n) \leq T(n)$$

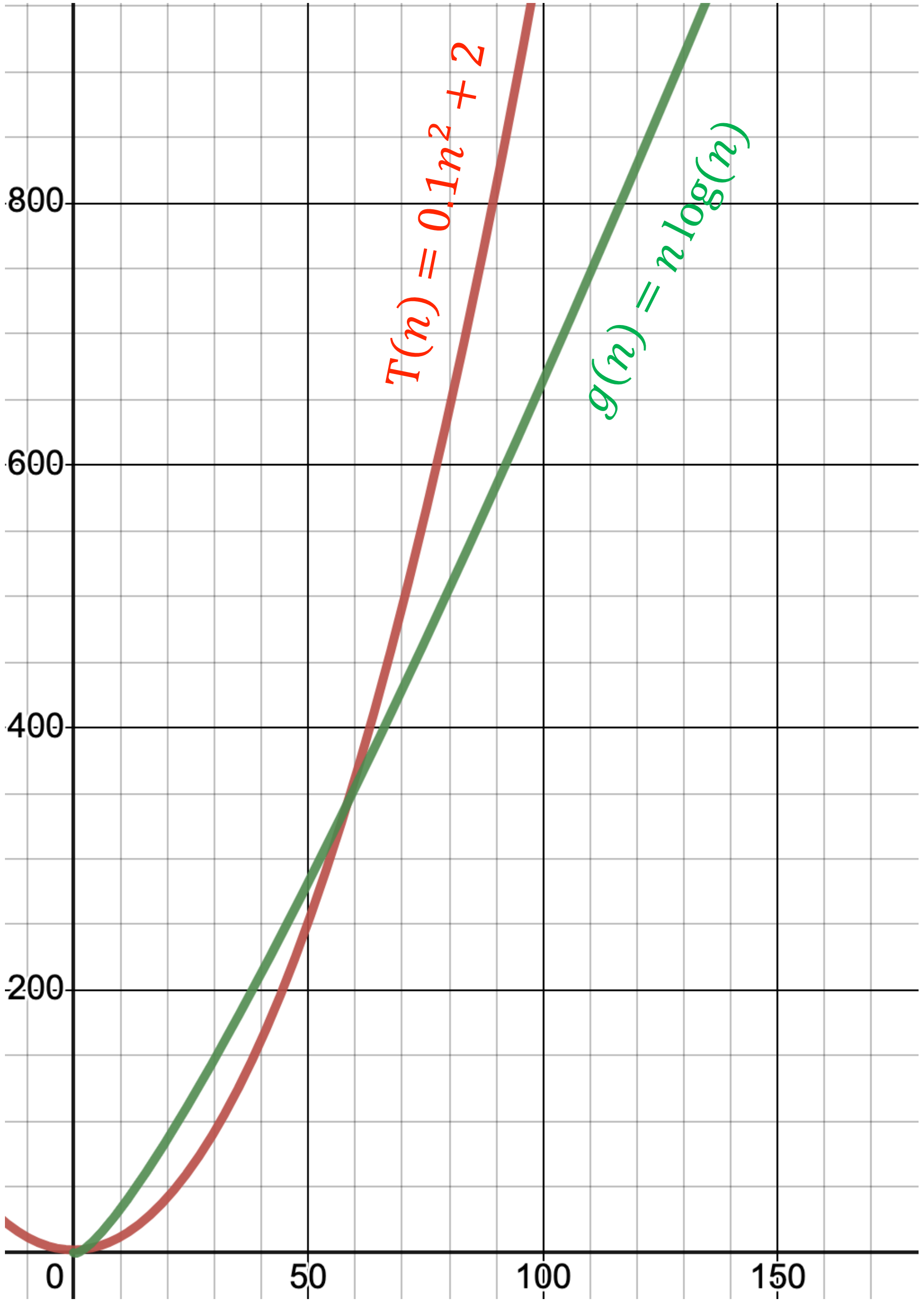
Switched these
compared to O( )!!

Example

Indeed, $0.1n^2 + 2 \in \Omega(n \log(n))$!

Prove this formally:

Find constants c and $n_0 > 0$, such that for all $n \geq n_0$, $c\, n \log(n) \leq 0.1n^2 + 2$.



$\Theta(\ldots)$ means both!

We say "$T(n)$ is $\Theta(g(n))$" iff both:

$$T(n) = O\big(g(n)\big)$$

and

$$T(n) = \Omega\big(g(n)\big)$$

Example: Asymptotics of the geometric series

Take any constant r and function $T(n) = 1 + r + r^2 + \cdots + r^n$

Show that $T(n) = \begin{cases} \Theta(r^n) & \text{if } r > 1 \\ \Theta(1) & \text{if } r < 1 \\ \Theta(n) & \text{if } r = 1 \end{cases}$

Prove formally at home!

Proof Idea: Recall sum of a geometric series that for $r \neq 1$:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Intuition:

- For $r > 1$, this is approximately $\frac{r^{n+1}}{r} = r^n$.
- For $r < 1$, $\frac{r^{n+1}-1}{r-1} \approx \frac{1}{1-r}$

Prove formally at home (also EX 0.2 of the book).

Revisiting Karatsuba's Alg runtime, more formally

What is the runtime of Karatsuba's Alg?

Karatsuba's Alg in 1 layer

Q1 $= a \times c$ Q2 $= b \times d$ Q3 $= (a+b)(c+d)$

$x \times y = \text{Q1} \times 10^n + (\text{Q3} - \text{Q1} - \text{Q2})10^{n/2} + \text{Q2}$

At each layer, we have 3 problems

→ Each problem of size $\frac{n}{2}$.

We have to do a bunch of other operations

- Finding a, b, c, d by shifting n-digit arrays.
- $n/2$-digit additions $a+b$, c+d
- n-digit additions $\text{Q3} - \text{Q1} - \text{Q2}$
- $2n$-digit additions $\text{Q1} \times 10^n + (\text{Q3} - \text{Q1} - \text{Q2})10^{n/2} + \text{Q2}$
- ....

$O(n)$

More precisely $\leq 20n$

Runtime : $$T(n) = \overbrace{3\,T\left(\frac{n}{2}\right)}^{\text{Recursive calls to the next layer}} + \overbrace{20n}^{\text{Work in this layer}}$$

Recurrence Relations

Recurrence relations give a formula for $T(n)$, i.e., the runtime on size n problems in terms of $T(k)$ where $k < n$.

$$\underbrace{3\,T\left(\frac{n}{2}\right)}_{\text{Recursive calls to the next layer}} \qquad \underbrace{20n}_{\text{Work in this layer}}$$

$T(n) = 3\,T\left(\frac{n}{2}\right) + 20n$ is a **recurrence relation.**

$T(1) = O(1)$ Base case (e.g., $T(1) = 5$ *or* 500)

Main question:

Given a recurrence relation for $T(n)$, find a closed-form expression for it.

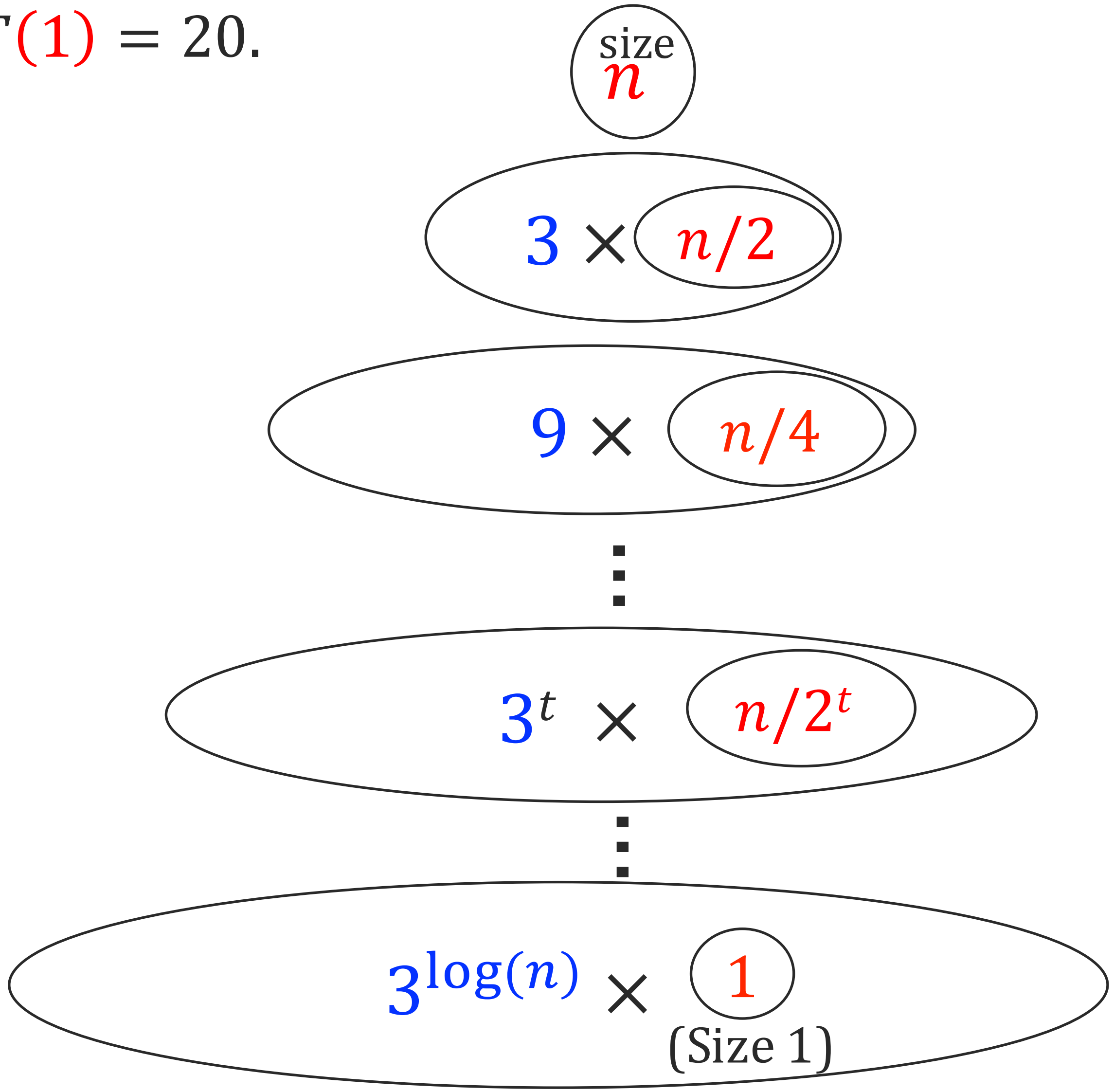
For example, we hope that $T(n) = O(n^{1.6})$ for the above recurrence!

Solve Karatuba's Alg Recurrence Relation

$$T(n) = 3T\left(\frac{n}{2}\right) + 20n, \quad T(1) = 20.$$

Abstraction of the tree

Total contribution in this layer



Solve Karatuba's Alg Recurrence Relation

$T(n) = 3T\left(\frac{n}{2}\right) + 20n, \quad T(1) = 20.$

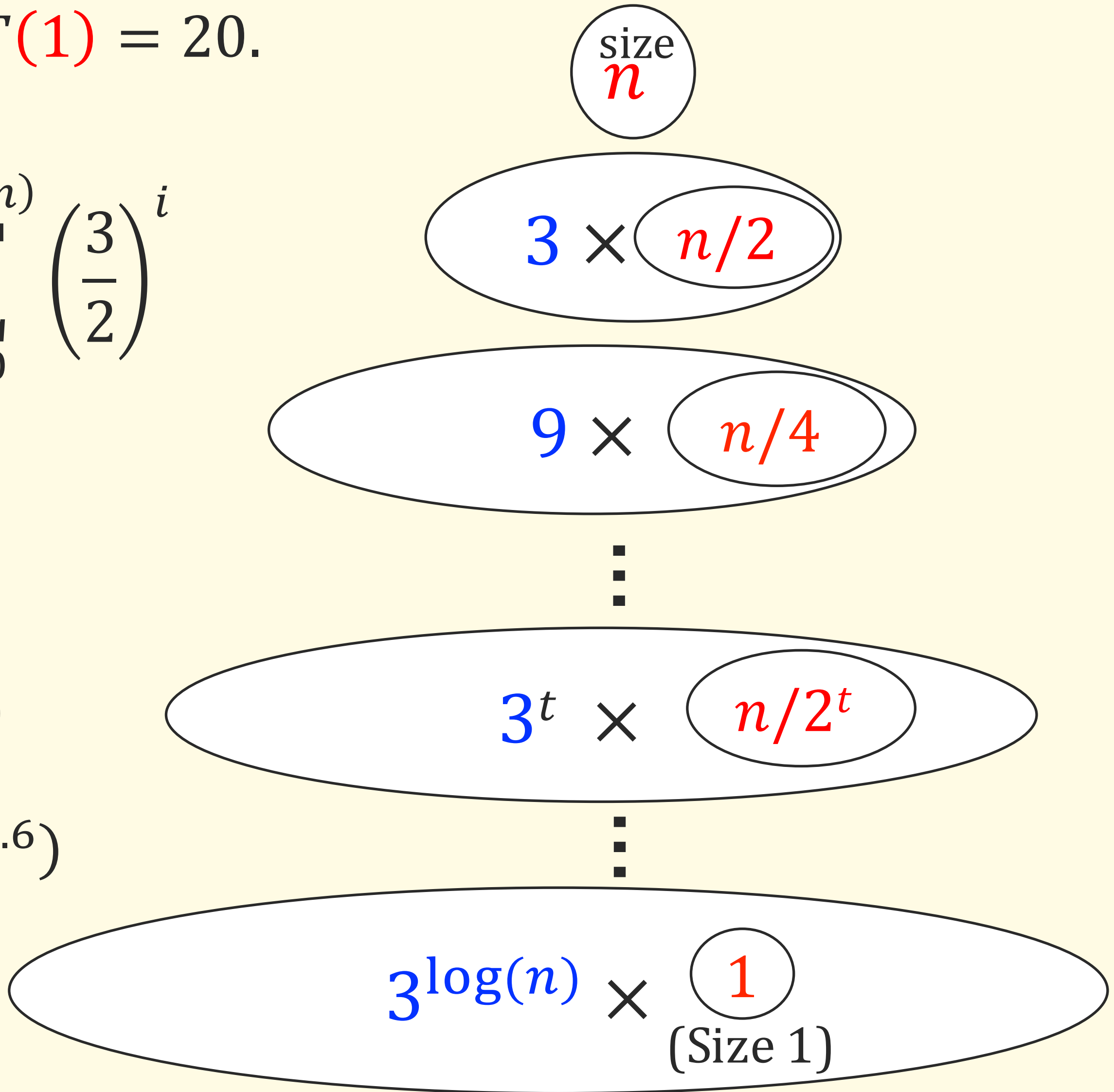
$$\sum_{i=0}^{\log(n)} 20n\left(\frac{3}{2}\right)^i = 20n\sum_{i=0}^{\log(n)}\left(\frac{3}{2}\right)^i$$

$$= O\left(n\left(\frac{3}{2}\right)^{\log(n)}\right)$$

$$= O\left(n \times n^{\log 3 - \log 2}\right)$$

$$= O\left(n^{\log(3)}\right) = O\left(n^{1.6}\right)$$

Abstraction of the tree



Total contribution in this layer

$20n$

$3 \times 20\left(\frac{n}{2}\right)$

$9 \times 20\left(\frac{n}{4}\right)$

$3^t \times 20\left(\frac{n}{2^t}\right)$

$3^{\log(n)} \times 20(1)$

Solving Recurrence Relations Generally

The tree method, as we just did

- Keep track of the number and size of problems in each step
- Account for total amount of computation done in each layer.
- Sum over all the computation done in the layers.

The Master Theorem

The tree method, as we just did

- Keep track of the number and size of problems in each step
- Account for total amount of computation done in each layer.
- Sum over all the computation done in the layers.

The Master Theorem

Suppose that $a \geq 1, b > 1$, and $d \geq 0$ are constants (independent of n).

Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

More on the Master Theorem

- Can it be used to solve any recurrence relation?

→ Nope! But it is a useful tool in many cases.

→ So, make sure you are also comfortable with the tree method.

- Don't we need a base case?

→Yes!

→Take $T(1) = O(1)$, the exact constant in this case doesn't affect the O(.).

- What if n/b is not an integer?

→The Master Theorem is also correct with $T(n) = a \cdot T\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$.

→We will mostly **ignore floors and ceilings** in recurrence relations.

Overview of the proof of Master Theorem

- See Section 2.2 of the book for a complete proof.

For the proof, suppose that $T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$.

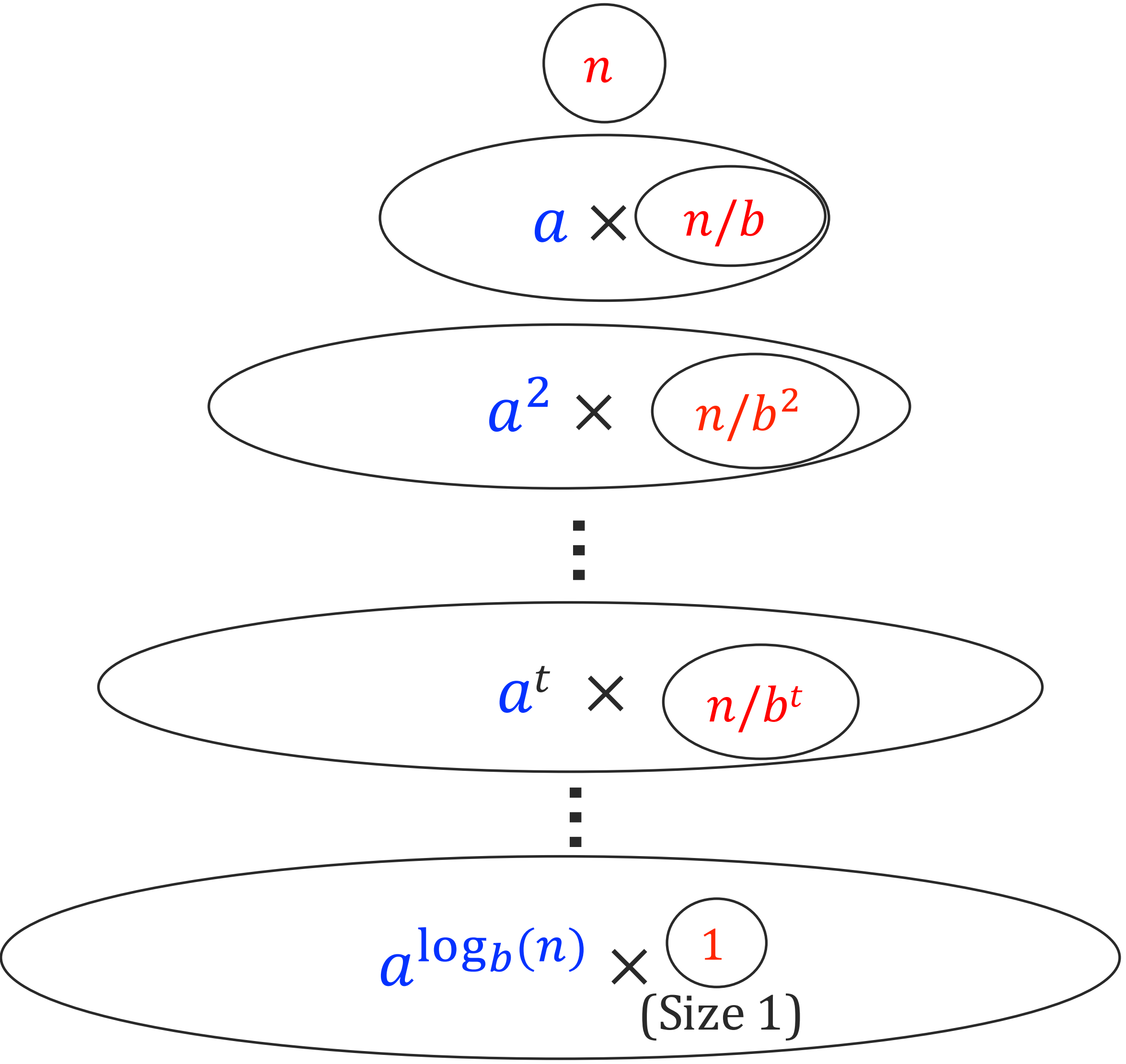
- For formal recursive arguments, we always substitute a constant.

→ Precise relationship between each layer's parameter and the amount of work.

→ Let's assume $T(1) = c$, too. For convenience!

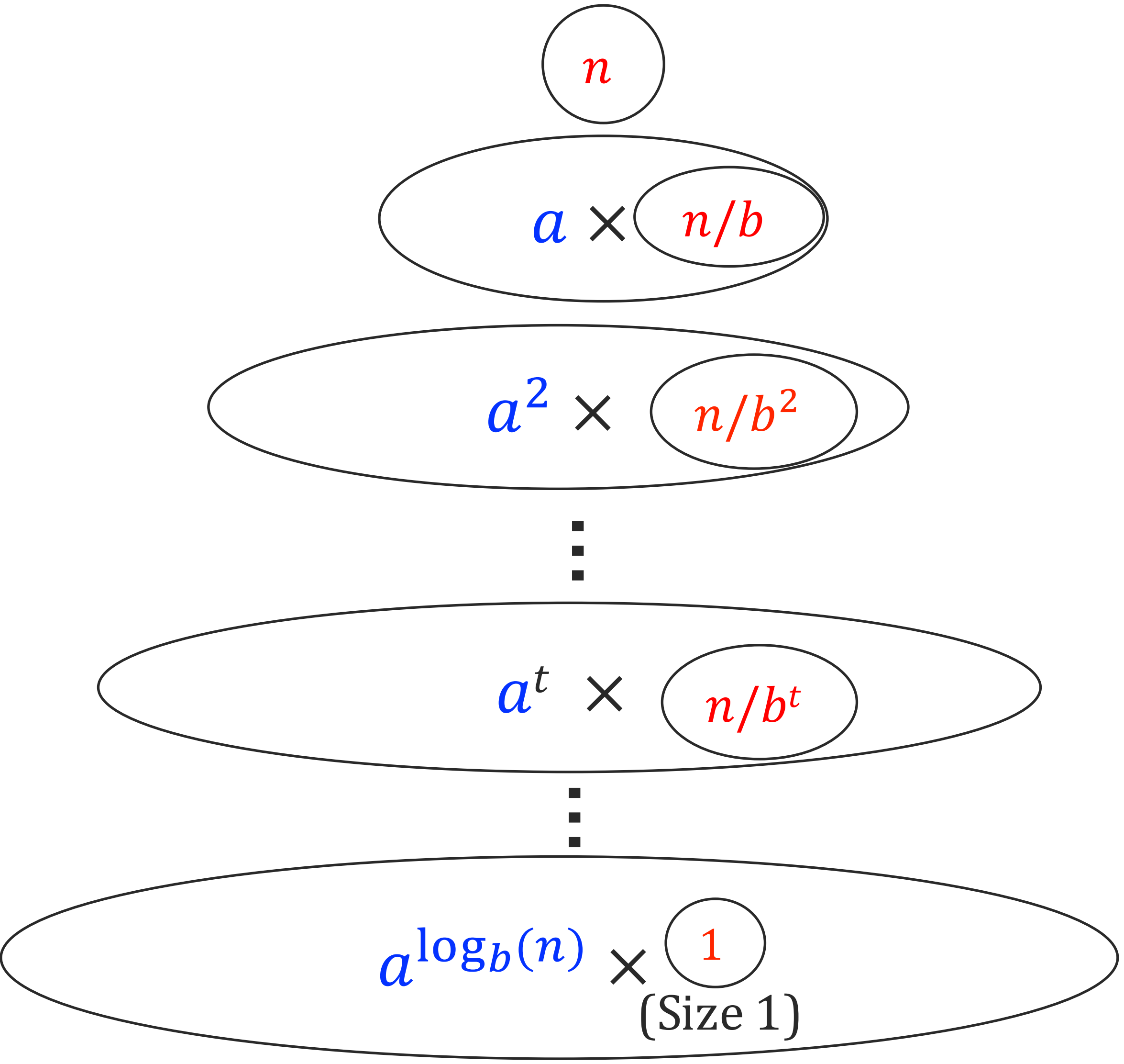
→ Just do the tree method!

$$T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



Layer	Problem size	# problems	Work @ this layer
0	n	1	
1	n/b	a	
$\vdots$	$\vdots$	$\vdots$	
t	n/b^t	a^t	
$\vdots$	$\vdots$	$\vdots$	
$\log_b(n)$	1	$a^{\log_b(n)}$	

$$T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



Layer	**Problem size**	**# problems**	**Work @ this layer**
0	n	1	$c \cdot n^d$
1	n/b	a	$a \cdot c \cdot \left(\frac{n}{b}\right)^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	n/b^t	a^t	$a^t \cdot c \cdot \left(\frac{n}{b^t}\right)^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_b(n)$	1	$a^{\log_b(n)}$	$a^{\log_b(n)} \cdot c$

$$T(n) \le a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Total computation on all layers:

$$cn^d \cdot \underbrace{\sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}_{\text{Looks so familiar ...}}$$

Layer	Problem size	# problems	Work @ this layer
0	n	1	$c \cdot n^d$
1	n/b	a	$a \cdot c \cdot \left(\frac{n}{b}\right)^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	n/b^t	a^t	$a^t \cdot c \cdot \left(\frac{n}{b^t}\right)^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_b(n)$	1	$a^{\log_b(n)}$	$a^{\log_b(n)} \cdot c$

Proof of the Master Theorem

$$T(n) \le a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Total computation on all layers:

$$cn^d \cdot \underbrace{\sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}_{\text{Geometric series}}$$

$$\boxed{1 + r + r^2 + \cdots + r^n = \begin{cases} \Theta(r^n) & \text{if } r > 1 \\ \Theta(1) & \text{if } r < 1 \\ \Theta(n) & \text{if } r = 1 \end{cases}}$$

The Master Theorem

$$T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

$$n^d \cdot n^{\log_b\left(\frac{a}{b^d}\right)} = n^{d+\log_b(a)-\log_b(b^d)} = n^{d+\log_b(a)-d} = n^{\log_b(a)}$$

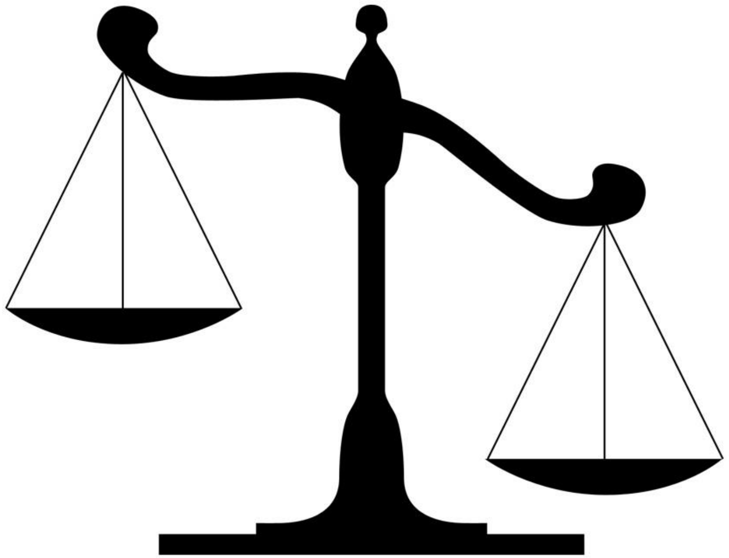
Total computation on all layers:

$$\underbrace{cn^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}_{\text{Geometric series}} = \begin{cases} \overbrace{\Theta\left(n^d \left(\frac{a}{b^d}\right)^{\log_b(n)}\right)} & \text{if } a > b^d \\ \Theta\left(n^d\right) & \text{if } a < b^d \\ \Theta\left(n^d \log(n)\right) & \text{if } r = 1 \end{cases}$$

$$\boxed{1 + r + r^2 + \cdots + r^n = \begin{cases} \Theta(r^n) & \text{if } r > 1 \\ \Theta(1) & \text{if } r < 1 \\ \Theta(n) & \text{if } r = 1 \end{cases}}$$

Master Theorem's Interpretation

a vs. b^d



Wide tree
$a > b^d$

Branching causes the number of problems to explode!
Most work is at the bottom of the tree!

Tall and narrow
$a < b^d$

Problem size shrinks fast, so **most work is at the top of the tree!**

$a = b^d$
Branching perfectly balances total amount of work per layer.
All layers contribute equally.

Wrap up

Karatsuba Integer Multiplication:

You can do better than grade school multiplication!

Example of divide-and-conquer in action

Runtime analysis, informal and formal.

Asymptotics, recurrence relations, and Master theorem

Tree method is intuitive and fun!

Master theorem is useful!

Next time

- More divide and conquer
- Matrix multiplications
- Median selection