

CS 170

Efficient Algorithms and Intractable Problems

Lecture 12

Dynamic Programming II

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Recap of the Last Lecture

Dynamic Programming!

The recipe!

Step 1. Identify subproblems (aka optimal substructure)

Step 2. Find a recursive formulation for the subproblems

Step 3. Design the Dynamic Programming Algorithm

→ Memo-ize computation starting from smallest subproblems and building up.

We saw a lot of examples already

→ Fibonacci

→ Shortest Paths with negative edge-weights (in DAGs, Reliable Shortest path, Bellman-Ford)

This lecture

See more dynamic programming examples!

- Shortest Path between all pairs
- Longest increasing subsequence
- Edit distance
- And even more next lecture

Best way to learn dynamic programming is by doing a lot of examples!

By doing more examples today, we will also develop intuition about how to choose subproblems (Recipe's step 1).

All-Pair Shortest Path Problem

All-Pair Shortest Path (APSP)

We want to know the shortest distance between any pair of nodes in a graph.

→ Not just from a special single source.

→ Another example of DP!

Input: Graph $G = (V, E)$, edge costs $\ell(u, v)$ for $(u, v) \in E$ (not necessarily positive)

Output: For all $u, v \in V$, $\text{dist}(u, v)$ = cost of shortest path from u to v

Naïve algorithm:

→ Run Bellman-Ford starting from every $s \in V$ as a source.

→ Bellman-Ford runs in time $O(mn)$

→ Total runtime for APSP would be $O(n^2m)$. Could be as large as $O(n^4)$ for dense graphs. We are aiming for $O(n^3)$.

Dynamic Programming Recipe

- **Step 1:** Identify the subproblems (optimal substructure)
- **Step 2:** Find a recursive formulation for the subproblems
- **Step 3:** Design the dynamic programming algorithm
 - Fill in a table, starting with the smallest sub-problems and building up.

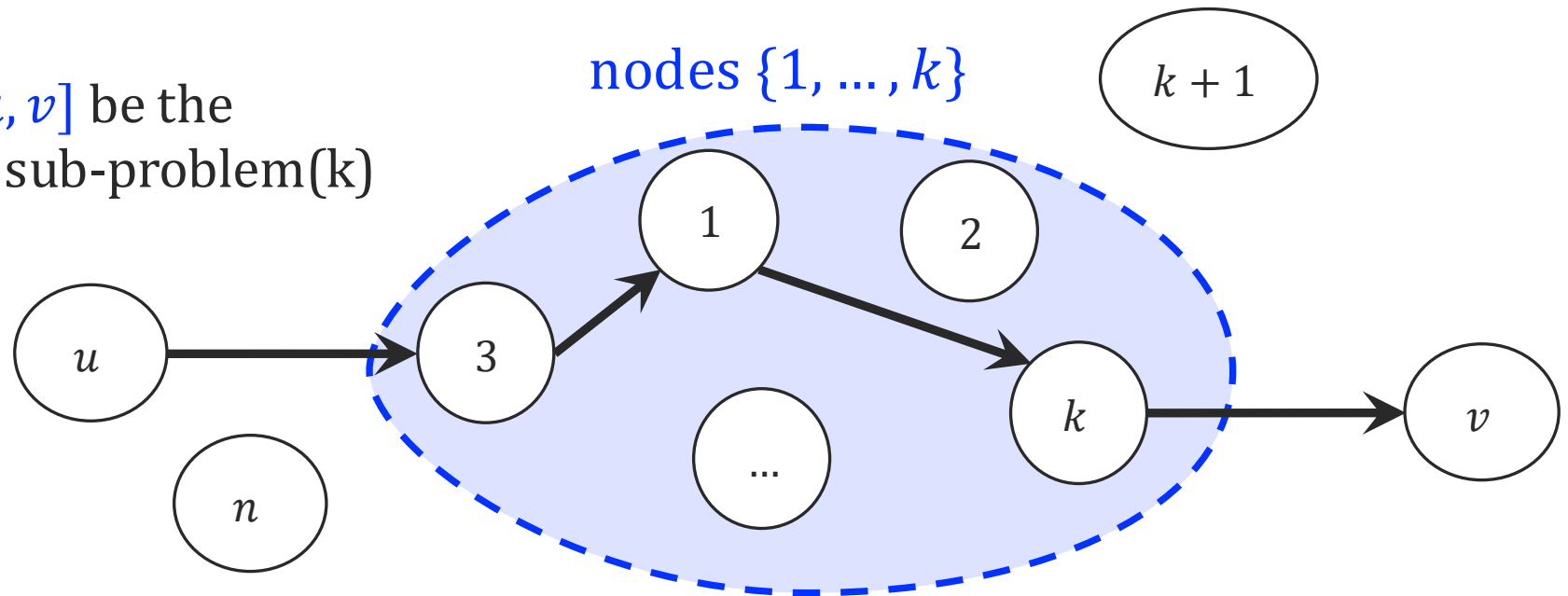
Identify the subproblems (optimal substructure)

Sub-problem(k): For all pairs $u, v \in V$, find the shortest u - v path whose internal vertices only use nodes $\{1, \dots, k\}$.

→ Sub-problem (n) is the APSP we want to solve.

→ This may look unintuitive, but let's see why it's helpful!

Let $dist_k[u, v]$ be the solution to sub-problem(k)



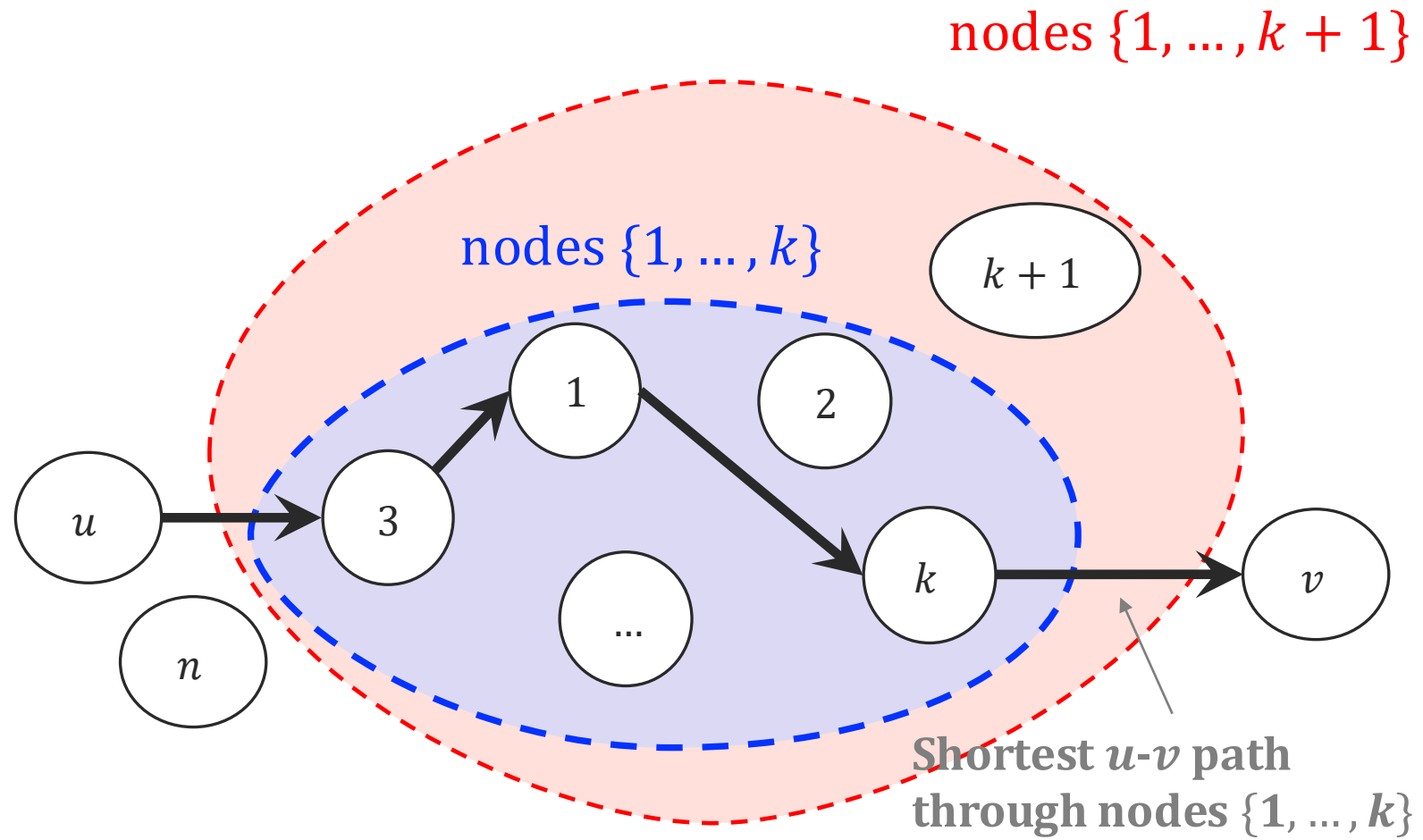
This is an overview picture, not all edges are shown.

Dynamic Programming Recipe

- **Step 1:** Identify the subproblems (optimal substructure)
- **Step 2:** Find a recursive formulation for the subproblems
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 - Fill in a table, starting with the smallest sub-problems and building up.

Recursive Formulation

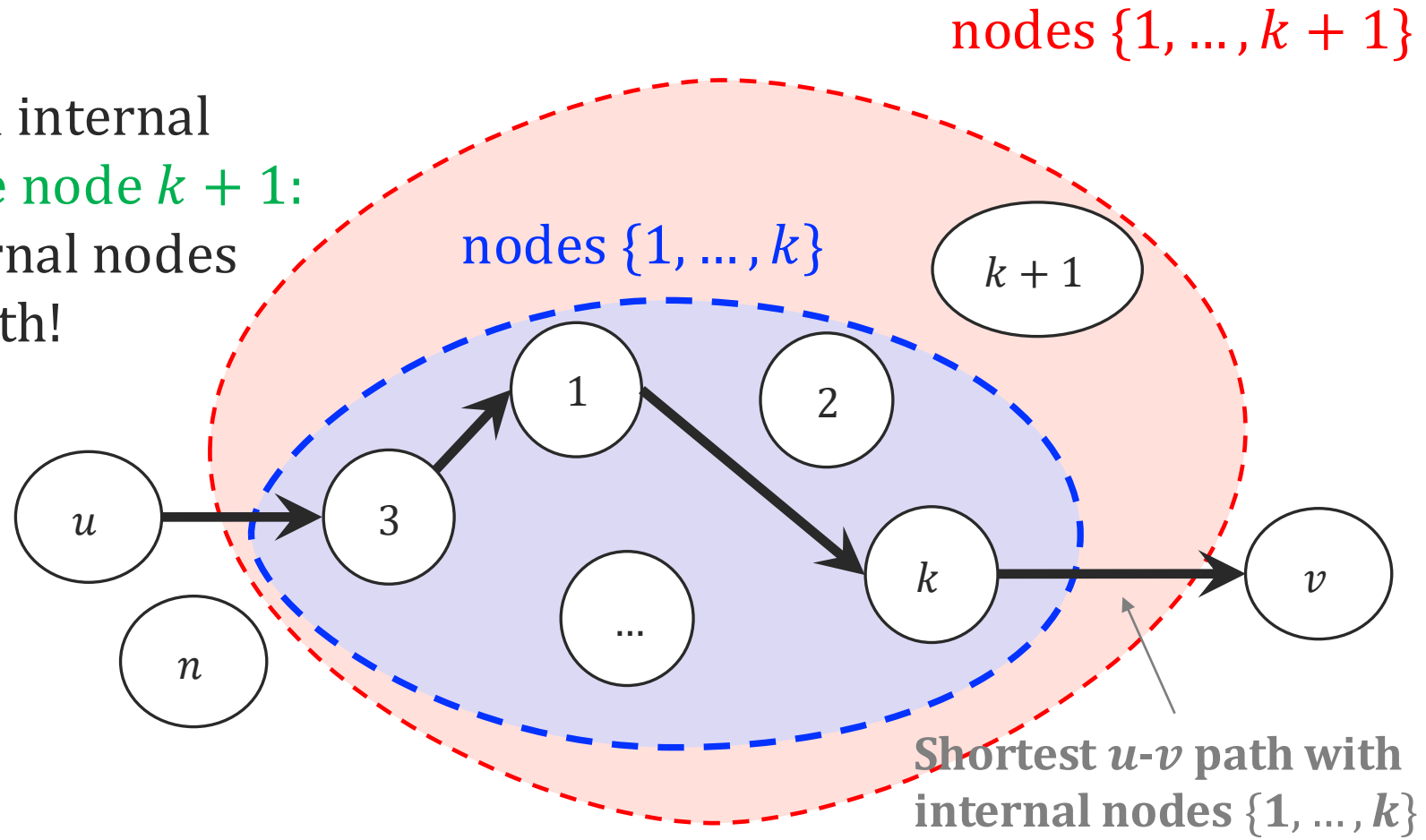
How do I solve **sub-problem(k+1)** knowing all the solutions $dist_k(u, v)$ to **Sub-problem(k)**?



Recursive Formulation

How do I solve **sub-problem(k+1)** knowing all the solutions $dist_k(u, v)$ to **Sub-problem(k)**?

Case 1: Shortest u - v path with internal nodes $\{1, \dots, k+1\}$ **doesn't use node $k+1$:**
→ Shortest u - v path with internal nodes $\{1, \dots, k\}$ is still the shortest path!
 $dist_{k+1}(u, v) = dist_k(u, v)$!



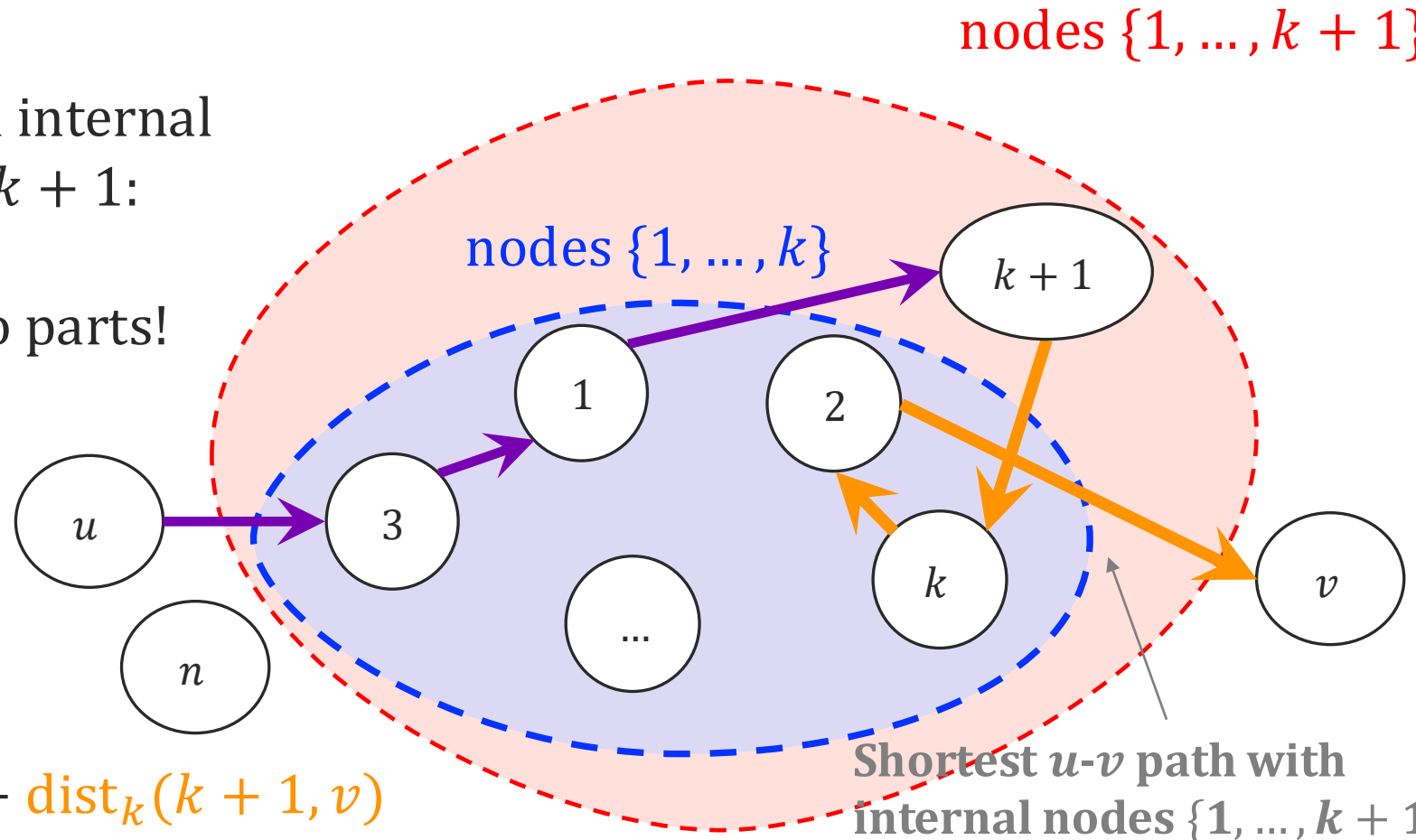
Recursive Formulation

How do I solve **sub-problem(k+1)** knowing all the solutions $dist_k(u, v)$ to **Sub-problem(k)**?

Case 2: Shortest u - v path with internal nodes $\{1, \dots, k+1\}$ uses node $k+1$:

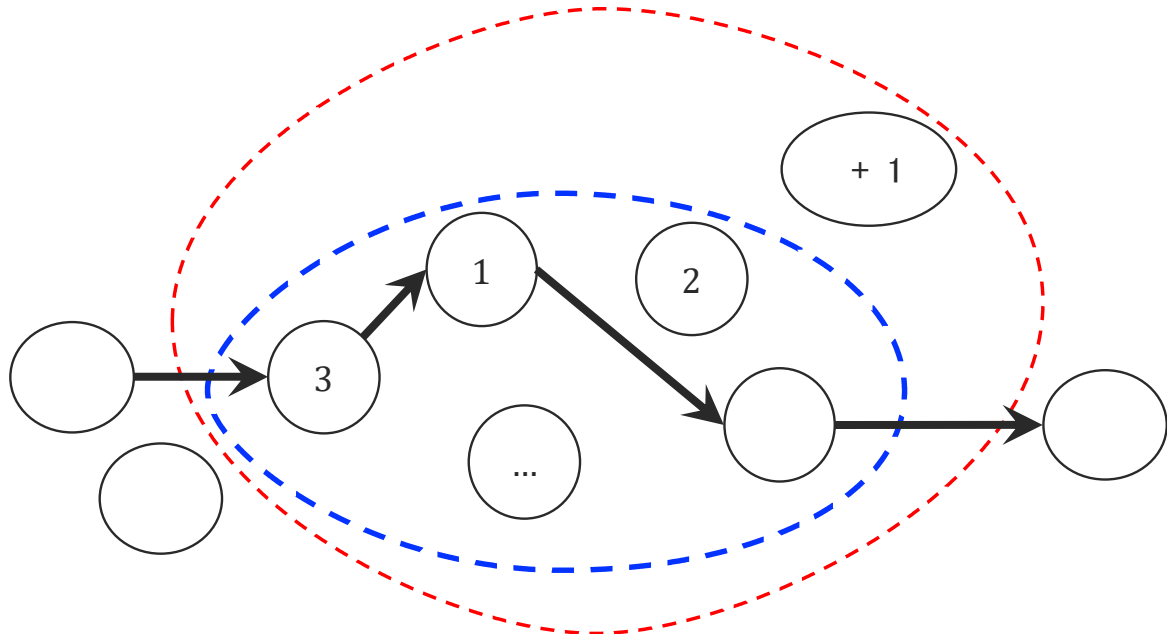
This path can be broken to two parts!

- Shortest u -($k+1$) path
- Shortest ($k+1$)- v path
- Both using **internal nodes** $\{1, \dots, k\}$ only.

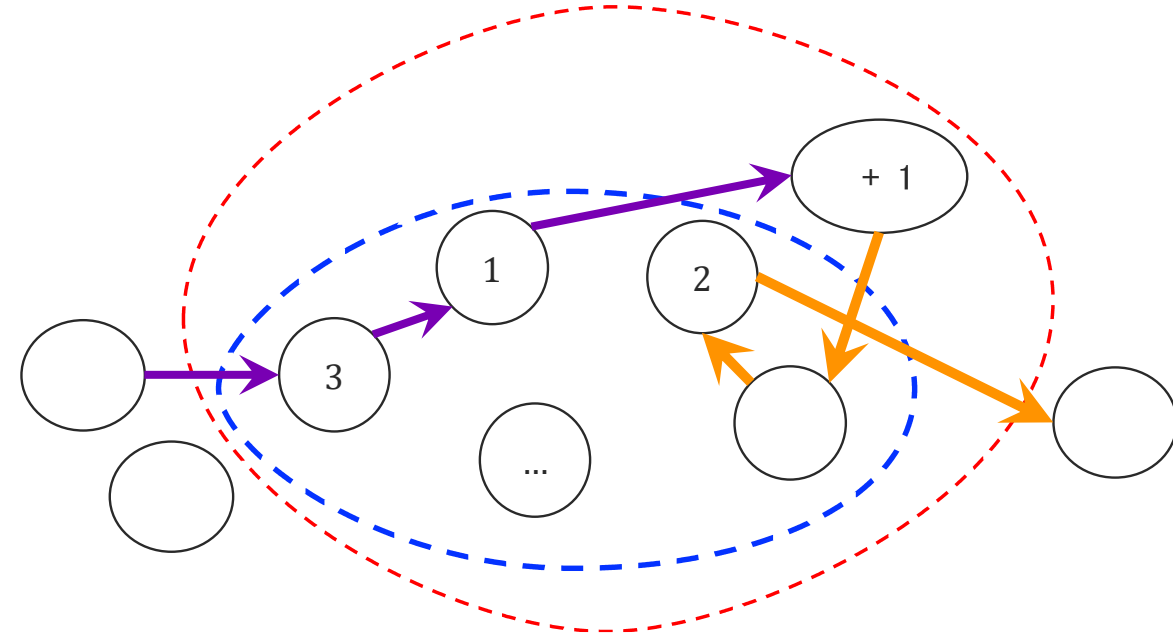


$$dist_{k+1}(u, v) = dist_k(u, k+1) + dist_k(k+1, v)$$

Putting the two cases together



Case 1: Shortest $u-v$ path with internal nodes $\{1, \dots, k + 1\}$ **doesn't use node $k + 1$:**



Case 2: Shortest $u-v$ path with internal nodes $\{1, \dots, k + 1\}$ **uses node $k + 1$:**

Discuss

Write the recursive relationship for the All-Pair Shortest Path problem:

Dynamic Programming Recipe

- **Step 1:** Identify the subproblems (optimal substructure)
- **Step 2:** Find a recursive formulation for the subproblems
- **Step 3:** Design the dynamic programming algorithm
 - Fill in a table, starting with the smallest sub-problems and building up.

The Floyd-Washall Algorithm for APSP

Input: Graph $G = (V, E)$, edge costs $\ell(u, v)$ for $(u, v) \in E$ (not necessarily positive)

Output: For all $u, v \in V$, $dist(u, v)$ = cost of shortest path from u to v

Each update is just $O(1)$.

The loop over k and u, v repeats $O(n^3)$ times.

Overall, $O(n^3)$ runtime.

Floyd-Warshall ($G = (V, E)$)

$n \times n$ matrices $dist_0, dist_1, \dots, dist_n$ initialized to ∞

For $(u, v) \in E$, $dist_0[u, v] \leftarrow \ell(u, v)$

// $dist_0$ paths have no internal nodes.

For $k = 1, \dots, n$:

For $u, v \in V$:

$dist_k[u, v] \leftarrow \min\{dist_{k-1}[u, v],$
 $dist_{k-1}[u, k] + dist_{k-1}[k, v]\}$

Longest Increasing Subsequences

Longest Increasing Subsequences (LIS)

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence of the input.

To be consistent with the book, we aren't using 0-indexing for the input.

$a = 5 \ 2 \ 8 \ 6 \ 3 \ 6 \ 9 \ 7$

“Subsequences” can be non-contiguous by definition.

Longest Increasing Subsequence

Why care about this problem?

- An important algorithmic preprocessing step.
- Useful for understanding random processes.
- Shuffle cards and play the game of Solitaire (aka Patience Sorting), how many piles you need?
- Computations over random graphs, networks, social media.

The recipe!

Step 1. Identify subproblems (aka optimal substructure)

Step 2. Find a recursive formulation for the subproblems

Step 3. Design the Dynamic Programming Algorithm

→ Memo-ize computation starting from smallest subproblems and building up.

Step 1: Subproblems of LIS

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence (LIS) of the input.

Discuss

Which of these two subproblems is more appropriate for designing a dynamic programming algorithm?

1. $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j], \text{ for } j = 1, \dots, n$

2. $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j] \text{ that ends in } a_j, \text{ for } j = 1, \dots, n$

What makes for good subproblems?

- Not too many of them (the more subproblems the slower the DP algorithm)
- Must have enough information in it to compute subproblems recursively (needed for step 2).

Step 1: Subproblems of LIS

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence (LIS) of the input.

Subproblems: $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j] \text{ that ends in } a_j$, for $j = 1, \dots, n$

- Because, if we don't keep track of the **last (largest) element of the LIS** we don't know **whether we can add a new element** to the subsequence, **recursively**.
- Think of the subproblem's stored info as the only thing you observe about smaller instances!

len. of LIS = 4

$a =$

5	2	8	6	3	6	9
---	---	---	---	---	---	---

 7

Knowing only Len of LIS, we
wouldn't know if we can add 7



Step 1: Subproblems of LIS

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence (LIS) of the input.

Subproblems: $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j] \text{ that ends in } a_j$, for $j = 1, \dots, n$

- Because, if we don't keep track of the **last (largest) element of the LIS** we don't know **whether we can add a new element** to the subsequence, **recursively**.
- Think of the subproblem's stored info as the only thing you observe about smaller instances!

len. of LIS = 4

$a = [5 \ 2 \ 8 \ 6 \ 3 \ 6 \ 9]$

len. of LIS ending in 9 = 4

Knowing only Len of LIS, we wouldn't know if we can add 7

We know for sure, we can't add 7



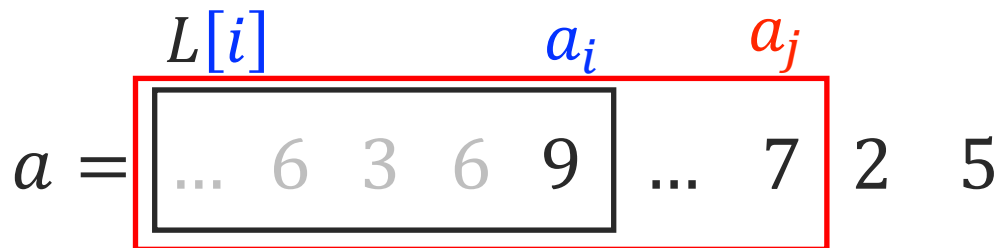
Step 2: Recurrence of LIS subproblems

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence (LIS) of the input.

Step 1: $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j] \text{ that ends in } a_j$

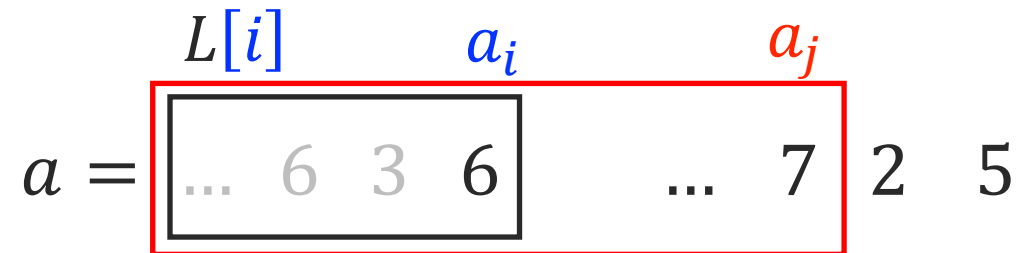
Step 2: Compute the recurrence: $L[j]$ in terms of $L[i]$ for $i < j$.

Case 1: $a[j] \leq a[i]$



Can't add $a[j]$ to
lengthen $L[i]$

Case 2: $a[j] > a[i]$



$$L[j] = L[i] + 1$$

Step 2: Recurrence of LIS subproblems

To be consistent with the book, we aren't using 0-indexing for the input.

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence (LIS) of the input.

Step 1: $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j] \text{ that ends in } a_j$

Discuss

Step 2: Compute the recurrence: $L[j]$ in terms of $L[i]$ for $i < j$.

Step 3: Design the DP Algorithm

- Input: An array of n integers $a = [a_1, \dots, a_n]$
- Output: The length of the longest increasing subsequence (LIS) of the input.

To be consistent with the book, we aren't using 0-indexing for the input.

Subproblems: $L[j] = \text{len. of LIS in array } [a_1, \dots, a_j] \text{ that ends in } a_j$

Runtime:

$O(n)$ subproblems

For each subproblem, we look at at most n smaller subproblems.

→ $O(n)$ time per subproblem.

Total: $O(n^2)$ runtime.

LIS($a_1, \dots, a_n$)

array L of length n

for $j = 1, \dots, n$

If exists $i < j$, s.t., $a_i < a_j$

$L[j] \leftarrow 1 + \max_{i < j} \{ L[i] \mid a_i < a_j \}$

Else $L[j] \leftarrow 1$

return $\max_i L[i]$

Edit Distance

Computing the Edit Distance

Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

Output: Compute the smallest number of edits to turn S into T .

Edits allowed:

1. Insert a character into S
2. Delete character from S
3. Change one character to another character.

Example:

What's the edit distance between
 $S = \text{"SNOWY"}$ and $T = \text{"SUNNY"}$?

S N O W Y

Add U

S U N O W Y

Change O to N

S U N N W Y

Delete W

S U N N Y

Applications of Edit Distance

- Auto correct!
- Word suggestions in search engines
- DNA analysis of similarities.

Edit Distance and Cost of Alignment

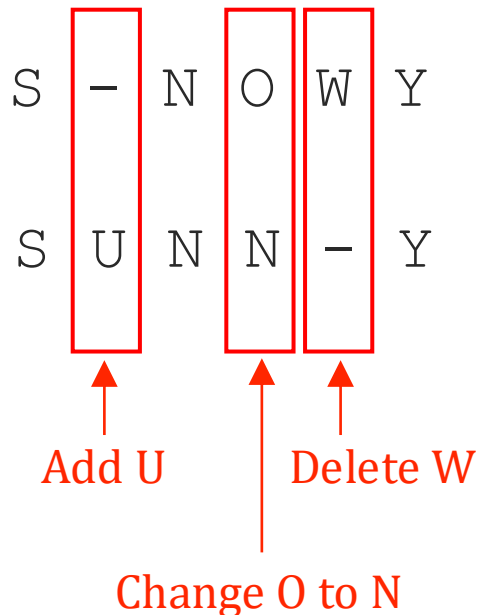
Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

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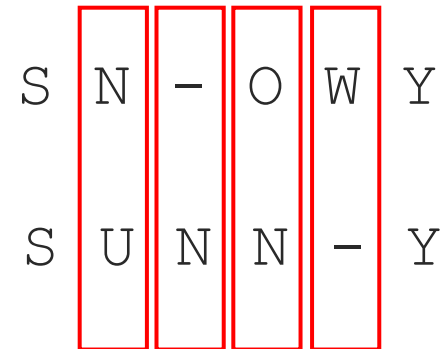
Edit Distance is the **minimal cost of alignment** between two strings.

→ **An alignment**: line up two words. **Cost of an alignment** = # columns that don't match

An alignment
with cost 3



An alignment
with cost 4



Step 1: Subproblems of Edit Distance

Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

Output: Compute the smallest number of edits to turn S into T .

What makes for good subproblems?

- Not too many of them (the more subproblems the slower the DP algorithm)
- Must have enough information in it to compute subproblems recursively (needed for step 2).

Subproblems: for all $0 \leq i \leq m$ and $0 \leq j \leq n$

$$E(i, j) = \text{EditDist}(S[1 \dots i], T[1 \dots j])$$

Cost of optimal alignment between $S[1 \dots i]$, $T[1 \dots j]$

Step 2: Recurrence Relation of Edit Distance

Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

Output: Compute the smallest number of edits to turn S into T .

Step 1: $E(i, j) = \text{EditDist}(S[1 \dots i], T[1 \dots j])$, for all $0 \leq i \leq m$ and $0 \leq j \leq n$

Discuss

There are three ways of aligning $S[1 \dots i]$ and $T[1 \dots j]$. What are their costs recursively?

Case 1

$S[1 \dots i - 1]$	$S[i]$
$T[1 \dots j]$	—

Case 2

$S[1 \dots i]$	—
$T[1 \dots j - 1]$	$T[j]$

Case 3

$S[1 \dots i - 1]$	$S[i]$
$T[1 \dots j - 1]$	$T[j]$

$E(i, j) =$

Step 2: Recurrence Relation of Edit Distance

Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

Output: Compute the smallest number of edits to turn S into T .

Step 1: $E(i, j) = \text{EditDist}(S[1 \dots i], T[1 \dots j])$, for all $0 \leq i \leq m$ and $0 \leq j \leq n$

Step 2: The recurrence relation

$$E(i, j) = \min \{E(i-1, j) + 1, E(i, j-1) + 1, E(i-1, j-1) + 1(S[i] \neq T[j])\}$$

Base case: $E(i, 0) = i$ and $E(0, j) = j$

Step 3: Design the Algorithm

Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

Output: Compute the smallest number of edits to turn S into T .

How do we memo-ize the subproblems in this recurrence relation?

$$E(i, j) = \min \{E(i-1, j) + 1, E(i, j-1) + 1, E(i-1, j-1) + 1(S[i] \neq T[j])\}$$

Base case: $E(i, 0) = i$ and $E(0, j) = j$

	0	...	$j-1$	j	...
0					
$\vdots$					
$i-1$			$E(i-1, j-1)$	$E(i-1, j)$	
i			$E(i, j-1)$	$E(i, j)$	
$\vdots$					

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$\vdots$					
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i			$E(i, j-1)$	$E(i, j)$	
$\vdots$					

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	0	...	$j-1$	j	...
0					
$\vdots$					
$i-1$			$E(i-1, j-1)$	$E(i-1, j)$	
i			$E(i, j-1)$	$E(i, j)$	
$\vdots$					

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Base case: $E(i, 0) = i$ and $E(0, j) = j$

	0	...	$j-1$	j	...
0					
$\vdots$					
$i-1$			$E(i-1, j-1)$	$E(i-1, j)$	
i			$E(i, j) - 1$	$E(i, j)$	
$\vdots$					

Runtime of this algorithm

Input: Two strings $S[1 \dots m]$ and $T[1 \dots n]$

Output: Compute the smallest number of edits to turn S into T .

$O(mn)$ number of subproblems.

For each subproblem, we take minimum of 3 values.

→ Work per subproblem $O(1)$

Total runtime: $O(mn)$.

```
Edit-Distance( $S[1 \dots m], T[1 \dots n]$ )
```

```
  ( $m + 1$ ) $\times$ ( $n + 1$ ) array  $E$ 
```

```
  For  $i = 0, 1, \dots, m, E[i, 0] = i$ 
```

```
  For  $j = 0, 1, \dots, n, E[0, j] = j$ 
```

```
  For  $i = 1, \dots, m$ 
```

```
    For  $j = 1, \dots, n$ 
```

```
       $E(i, j) \leftarrow \min \left\{ \begin{array}{l} E(i - 1, j) + 1, \\ E(i, j - 1) + 1, \\ E(i - 1, j - 1) + 1(S[i] \neq T[j]) \end{array} \right\}$ 
```

```
  return  $E(m, n)$ 
```

Wrap up

More examples of dynamic programming.

- Longest increasing subsequence
- Edit distance
- Knapsack (with repetition)

→ Also got more experience on how to choose subproblems.

Next time: More examples of DP
Knapsack and other graph problems