

Linear

Programming

John's bakery

Menu: Donuts \$5
Cakes \$25

Recipes

Ingredients	Donuts	Cake
Flour (200 total)	2	5
Sugar (300 total)	2	9
Eggs (500 total)	7	12

Goal: How many cakes & donuts to make to maximize profit?

Linear program (LP)

Decision variables:

$x = \#$ of donuts
 $y = \#$ of cakes (in $\mathbb{R}$)

Constraints: (Linear)

$$x \geq 0, y \geq 0$$

$$2x + 5y \leq 200$$

$$2x + 9y \leq 300$$

$$7x + 12y \leq 500$$

x and y are integers

Not allowed!

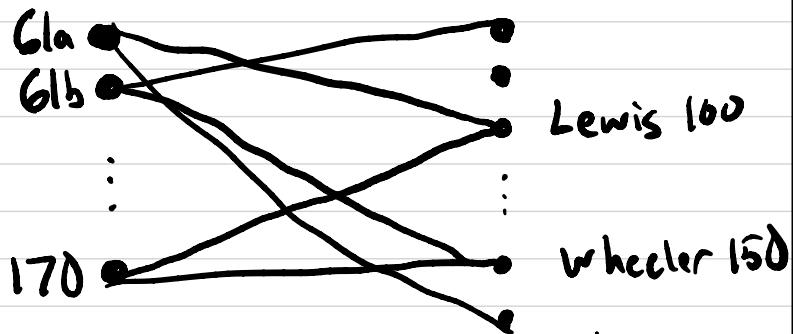
Gives Integer LP

Objective: maximize $5x + 25y$
(Linear)

Classroom allocation

Courses

Classrooms



Class c can be assigned room r

$$(c, r) \in E$$

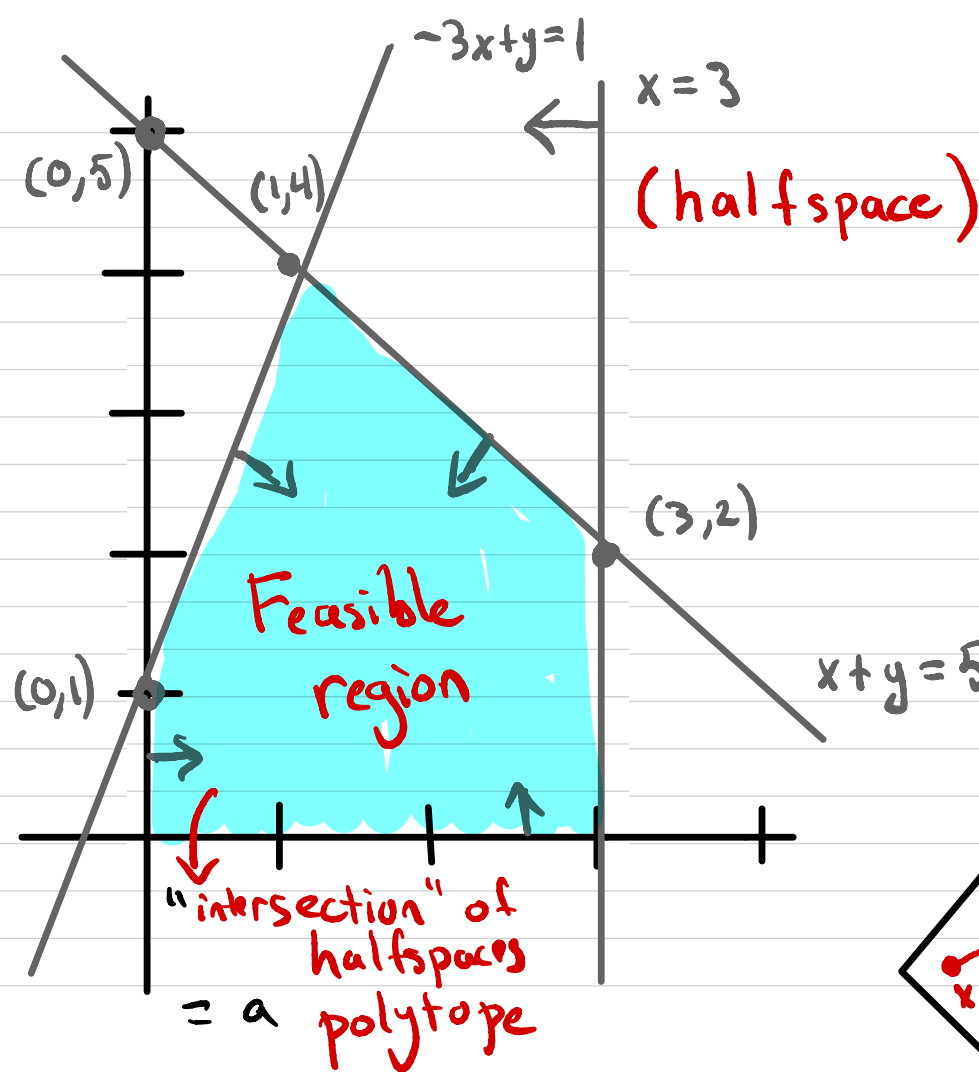
Goal: Maximize number of
courses assigned
classrooms

Linear Program (LP)

Variables

Constraints

Objective:

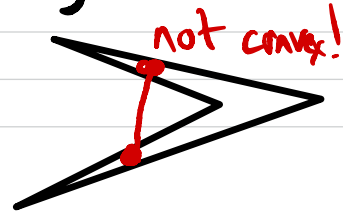
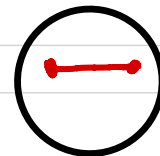
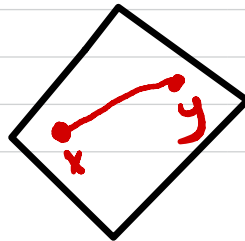


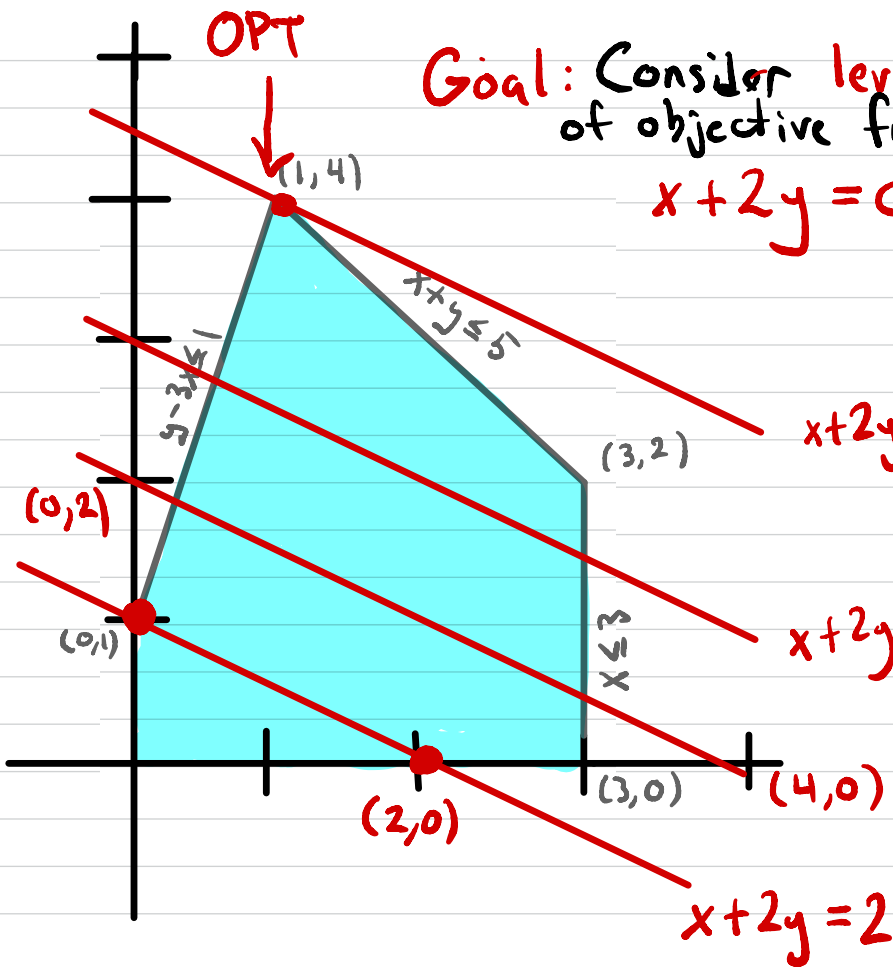
LP: $\max x+2y$
 subject to $x \leq 3$
 $x+y \leq 5$
 $-3x+y \leq 1$
 $x \geq 0$
 $y \geq 0$

Def: The **feasible region** is the set of points satisfying all constraints

Fact: The feasible region is **convex**

Def: A set of points $S \subseteq \mathbb{R}^d$ is **convex** if $\forall x, y \in S$, line segment $x \rightarrow y$ is in S .





Goal: Consider level sets
of objective function
 $x + 2y = c$

$$\begin{aligned} \text{LP: } \max \quad & x + 2y \\ \text{subject to} \quad & x \leq 3 \\ & x + y \leq 5 \\ & -3x + y \leq 1 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

$$x + 2y = 9$$

Fact: For all LPs, there is an optimal solution at a **vertex**

$$x + 2y = 6$$

Fact: Can prove $\text{OPT} = 9$.

Pf: Any feasible point satisfies

$$\begin{aligned} & (x + y \leq 5) \cdot \frac{7}{4} \\ & + (-3x + y \leq 1) \cdot \frac{1}{4} \\ \hline & x + 2y \leq 9. \quad \square \end{aligned}$$

"LP duality"

General LP

Variables:

$$x_1, \dots, x_n$$

Constraints

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \end{array} \right\} m$$

Objective

$$\text{Maximize } c_1x_1 + \dots + c_nx_n$$

Input: a_{ij} 's
 b_i 's
 c_i 's

Variants

$$\sum_i a_i x_i \geq b_i$$

$$\rightarrow \sum_i -a_i x_i \leq -b_i$$

$$\sum_i a_i x_i = b_i$$

$$\rightarrow \left\{ \begin{array}{l} \sum_i a_i x_i \leq b_i \\ \sum_i a_i x_i \geq b_i \end{array} \right.$$

$$\text{Minimize } \sum_i c_i x_i$$

$$\rightarrow \text{Maximize } \sum_i -c_i x_i$$

General LP

Variables

$$x_1, \dots, x_n \in \mathbb{R}$$

Constraints

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \end{aligned} \right\} m$$

Objective

$$\text{Maximize } c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Matrix form

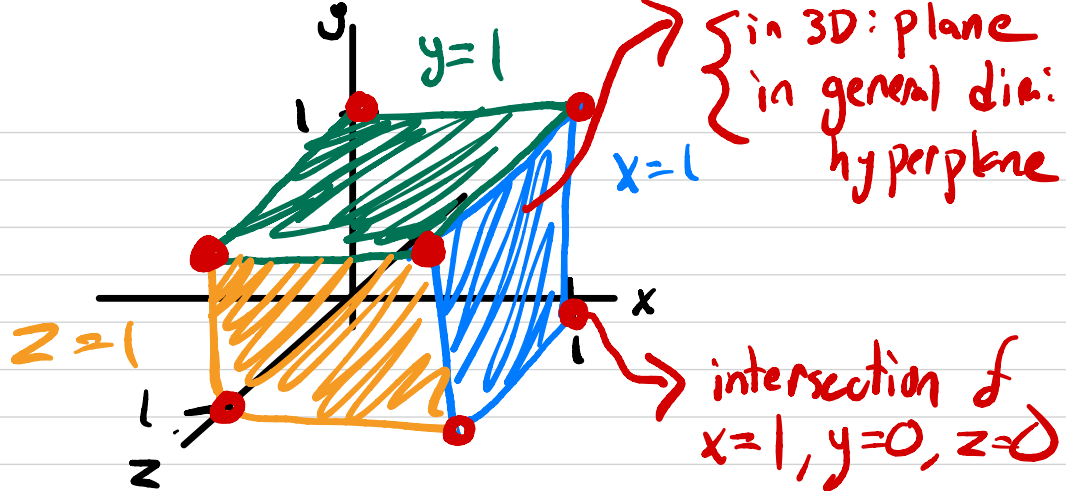
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$\text{LP} = \text{maximize } c^T x \\ \text{subject to } Ax \leq b$$

LP example

$$\begin{array}{ll}\max & x + y + z \\ \text{s.t.} & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ & 0 \leq z \leq 1\end{array}$$



Def: A **vertex** is a point x in the polytope ($x \in$ feasible set) which lies at the intersection of n hyperplanes in n dimensions

Recall: Always an optimal solution at a vertex

Fact: Suppose LP has n vars and m constraints.

Then # of vertices $\leq$ _____.

Linear Programming Alg 1: "Try all vertices"

Given m constraints $C = \{ \sum a_{ij} x_j \leq b_j \}$

For each subset $S \subseteq C$ of size n :

1. Solve for the point of intersection x^*

(solving linear system / Gaussian elimination)

2. Check if x^* is feasible (satisfies all constraints)

3. If so, compute value of x^*

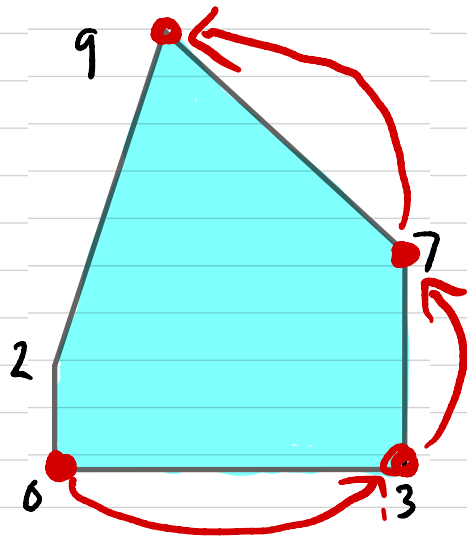
Output **best** vertex found.

Time complexity $\approx \binom{m}{n} \approx$ exponentially large in n

LP alg 2: Simplex

1. Start at a vertex x^*
2. Look at all "neighboring" vertices
3. Find the neighbor y^* with best value
4. If y^* has a better val than x^* , move to y^* . Repeat.

$n=2$:



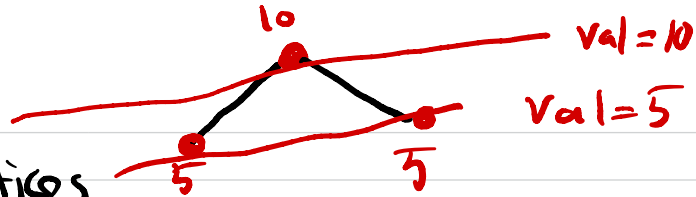
$n=3$: Constraints $\{A, B, C, D, E\}$

Vertex: (ABC)

Neighbors: $ABD, ABE, ACD, ACE, BCD, BCE$

Fact: Suppose n vars, m constraints
of neighbors $\leq$

Sad fact: Simplex still **exponential time**
(in the worst case)



Thm: Linear programs can be solved
in polynomial time.

[Khachiyan 1979]: Ellipsoid algorithm

[Kar markar 1984]: Interior points
algorithm