LECTURE PLAN

1) REDUCTIONS (RECAP)

2) Reductions Example : INDSET $\leq_p$ INT. PROG

3) NP- Completeness

4)

P = class of problems for which

CAN FIND solution in polynomial time

NP = class of problems for which

CAN VERIFY A GIVEN SOLUTION

in polynomial time.

REDUCTIONS

$$\underset{\text{(Max Matching)}}{\text{PROBLEM } A} \leq_p \underset{\text{(Max Flow)}}{\text{PROBLEM } B}$$

if "AN ALGORITHM TO SOLVE B (in poly time) CAN BE USED TO SOLVE A"

$\Rightarrow$ PROBLEM B is AT LEAST AS HARD as PROBLEM A (up to polynomial factors)

$\Rightarrow$ "$B \in P \Rightarrow A \in P$"

HOW TO SHOW $A \leq_p B$

1) REDUCTION ALGORITHM (F)

$F : \begin{pmatrix} \text{INPUT to} \\ \text{PROBLEM A} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{INPUT to} \\ \text{PROBLEM B} \end{pmatrix}$

2) PROOF: $\Longleftrightarrow$

$\exists$ a solution to $I \in A$ $\Longrightarrow$ $\exists$ a solution to $F(I) \in B$

AND

$\exists$ a solution to $F(I) \in B$ $\Longrightarrow$ $\exists$ a solution to $I \in A$.

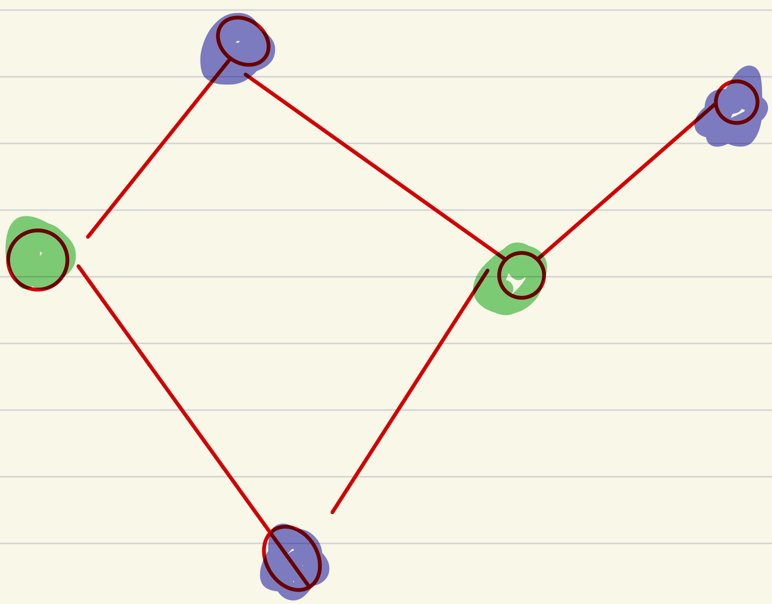
IND-SET:

INPUT: 1) GRAPH $G=(V,E)$

2) INTEGER: $K \in \mathbb{N}$

SOL: An independent set of size $\geq K$

A set $S \subseteq V$ is independent if there ARE NO EDGES INSIDE S.



INTEGER PROGRAMMING:

INPUT: 1) A linear programming over variables $\{x_1, \dots, x_n\}$

2) Bound B

SOL: A feasible INTEGER solution with value $\geq B$

$$\text{Max } x_1 + x_2 + 2x_3$$

$$x_1 + 3x_2 \leq 5$$

$$x_2 + 4x_3 \leq 7$$

$$x_1, x_2, x_3 \geq 0$$

IND SET

INPUT: Graph $G=(V,E)$, integer K

SOL: IND-SET of size K

$$\leq_p$$

INTEGER PROGRAMMING

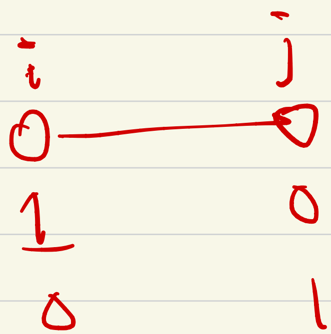
INPUT: An LP over $\{x_1, \dots, x_n\}$, B

SOL: Integer solution of value $\geq B$

$\boxed{\text{REDUCTION ALG } (F)}$

F: Graph $G=(V,E)$ $\longrightarrow$

Size: K



$x_1, \dots, x_n \in \{0,1\}$ be the variables

$x_i = 1$ if $i \in$ IND SET

$$\text{Max} \sum_{i=1}^{n} x_i = \text{Size of ind set}$$

$\forall$ edge $(i,j) \in E$

$$x_i + x_j \leq 1$$

$$0 \leq x_i \leq 1$$

Bound $= K$

PROOF: G has ind set I of size k

$\Longrightarrow$ IP has solution of value k

BECAUSE: consider $x_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$

observe that $\sum x_i = |I| = k$

AND $\forall \{i,j\} \in E \quad x_i + x_j \leq 1$

IP has solution X of value k

$\Longrightarrow$ G has ind set of size k.

BECAUSE: Each $x_i \in \mathbb{Z}$ AND $0 \leq x_i \leq 1$

$\Longrightarrow$ $x_i = 0$ or 1.

Define $I = \{ \text{vertex}_i \mid x_i = 1 \}$

$$|I| = \sum_{i=1}^{n} x_i = K$$

AND we know $\forall$ edge $(i,j) \in E$

$$x_i + x_j \leq 1$$

$\Rightarrow$ BOTH x_i AND x_j are $\neq 1$

$\Rightarrow$ either $i \in I$ or $j \in I$ but NOT BOTH.

or NONE

I is an independent set

NP- Completeness:

NP- complete problems:

"Problems in NP such that every problem in NP reduce to them"

Problem A is NP-complete

if 1) $A \in NP$

2) $\forall\ B \in NP$

$$B \leq_p A.$$

Theorem: [COOK][KARP] [69-71] 3SAT, 3-COLORING, INT-PROG
.... are NP-complete.

By now, tens of thousands of
NP-complete problems

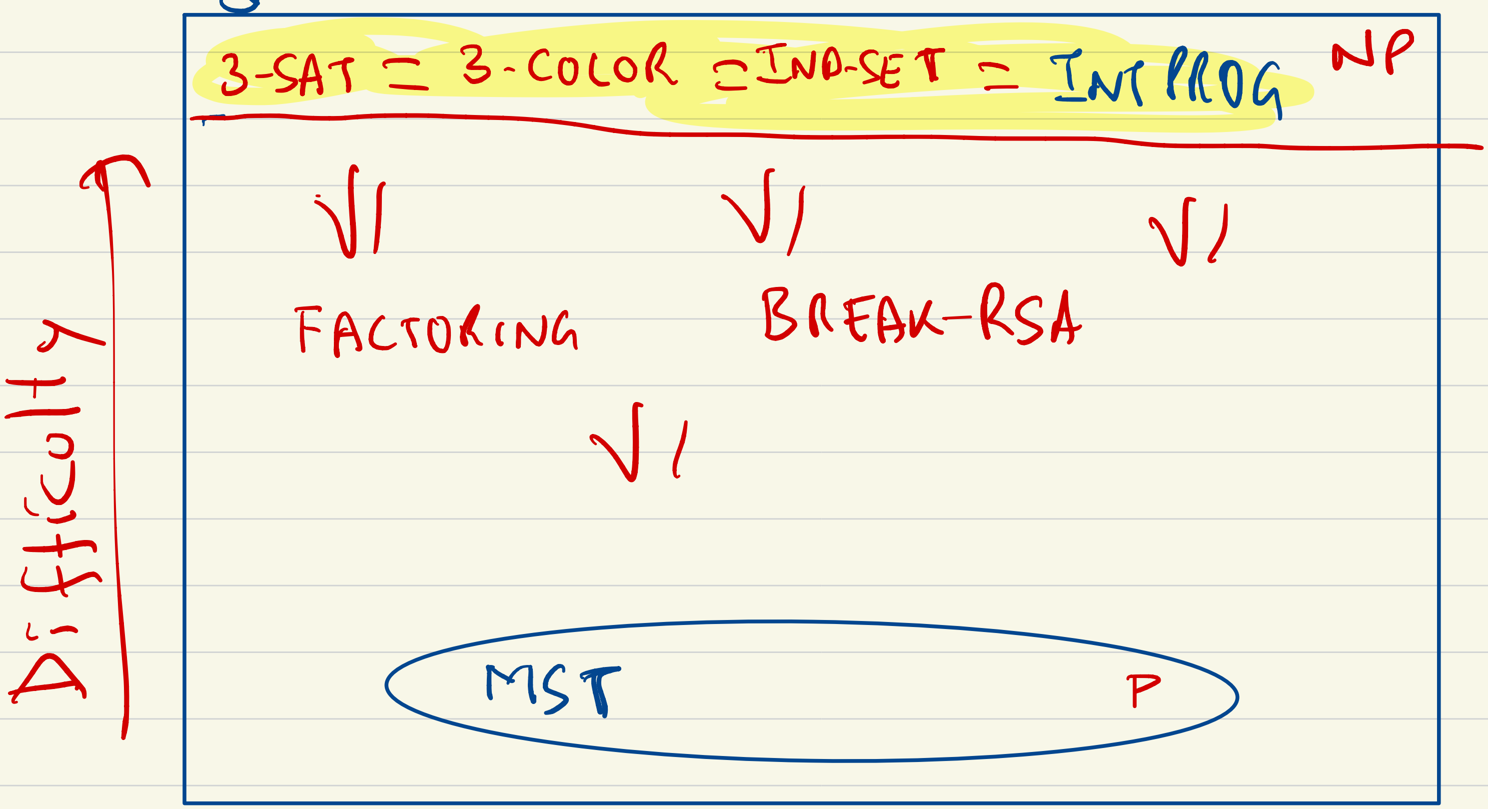
How to show that "A" is NP-complete

1) Prove $A \in NP$ ←

2) Take ANY known NP-complete problem B

$B \leq_p A$.

NP-complete problems



$A \longrightarrow B \iff A \leq_p B$

$$
\begin{tikzcd}[column sep=small]
 & & \text{“ALL OF NP”} \arrow[d, "{\text{[Cook70] (Cook's Theorem)}}"] & \\
 & & \text{CIRCUIT SAT} \arrow[d] & \\
 & & \text{3SAT} \arrow[dl] \arrow[dr] & \\
 & \text{IND SET} \arrow[dl] \arrow[d] \arrow[dr] & & \text{RUDRATA CYCLE (directed)} \\
\text{INTEGER PROGRAM} & \text{CLIQUE} & \text{VERTEX COVER} &
\end{tikzcd}
$$

3SAT:

INPUT: 1) Boolean Variables $x_1, \dots, x_n$

2) Clauses

$$\left.\begin{array}{l}(x_1 \vee \bar{x}_2 \vee x_7) \wedge \\ (x_5 \vee x_8 \vee \bar{x}_9) \wedge \\ \vdots \\ (\bar{x}_{100} \vee x_n \vee x_8)\end{array}\right\} m$$

SOL: An assignment satisfying all the clauses

3SAT:

INPUT: 3SAT FORMULA on $x_1 \dots x_n$

SOL: A satisfying assignment

P

IND SET

INPUT: Graph $G=(V,E)$, K

SOL: An independent set of size $\geq K$

REDUCTION ALG:

3SAT Formula $\Longrightarrow$ Graph G

1) $\forall$ clause $(x \vee y \vee \bar{z}) \Rightarrow$ add a Δ'le with vertices labelled with literals

2) $\forall$ variable x $\forall$ pair of nodes labelled x, $\bar{x}$ = ADD an edge

Set $K =$ # of clauses

3SAT Formula

$\{(\overset{1}{x} \vee \overset{0}{y} \vee \overset{0}{z})\} \wedge$

$\{(\bar{z} \vee \bar{w} \vee x)\} \wedge$

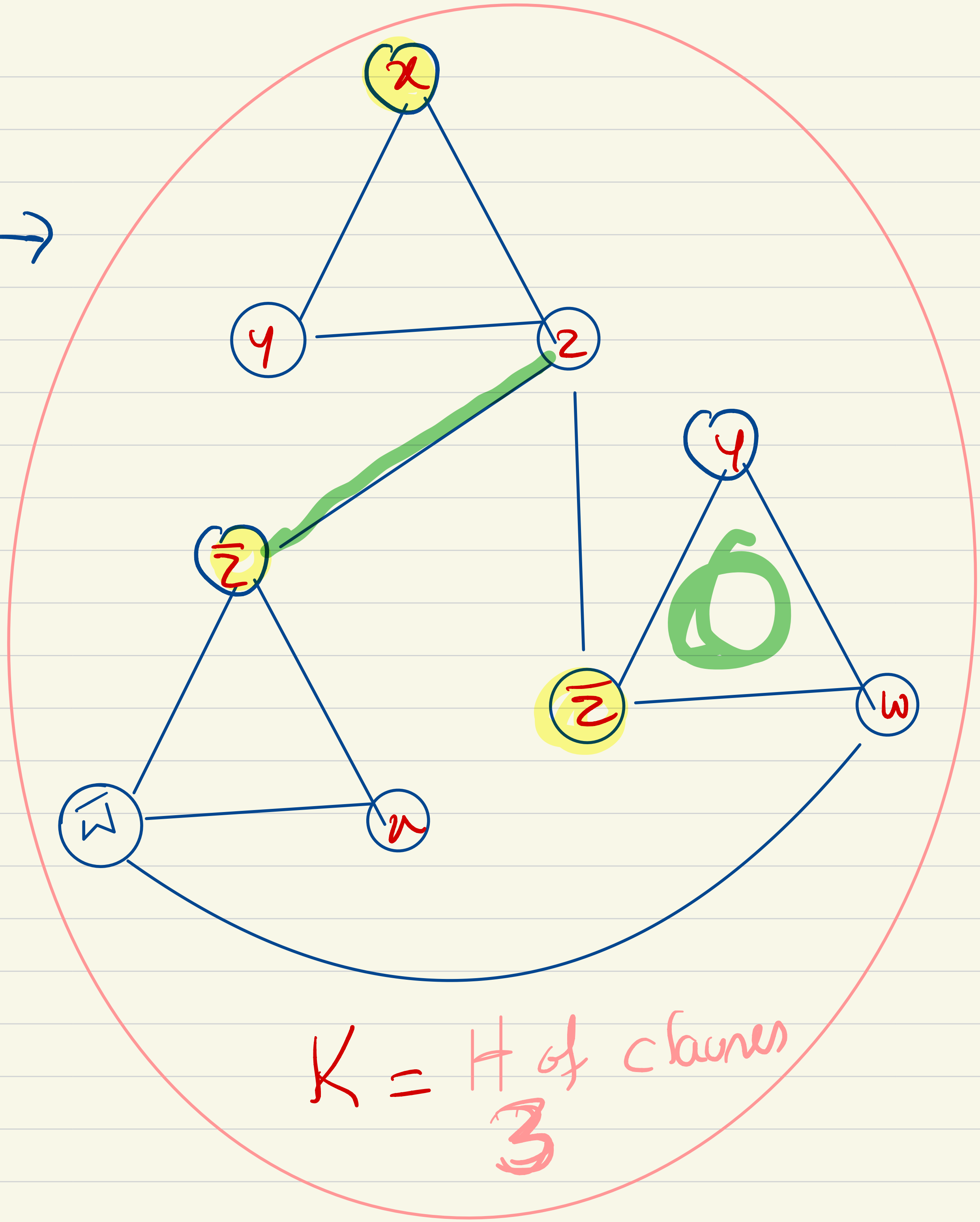
$\{(y \vee \bar{z} \vee w)\}$

$\boxed{x = 1 \quad y = 0 \quad z = 0 \quad w = 0}$

INTENTION:

$x = 1$

$\Longleftrightarrow$ Vertex x is in independent set



PROOF:

$\exists$ a 3SAT solution ($x_1 \ldots x_n \in \{0,1\}$) $\Longrightarrow$ $\exists$ an ind set of size K

$K = \#$ of clauses

BECAUSE: Given $x_1, \ldots, x_n \in \{0,1\}$
a satisfying assignment

In every clause $(x_i \vee \overline{x_j} \vee x_k) = 1$
at least one of x_i, $\overline{x_j}$, x_k is 1.

Add one of the vertices for true literals to ind set

$\Rightarrow$ gives an ind set of size

$= \#$ clauses

$\exists$ an independent set of size K

$\Rightarrow \exists$ a satisfying assignment

BECAUSE: $K = \#$ of clauses $= \#$ of $\triangle$'s

$\Rightarrow$ (Ind set picks exactly one vertex in every $\triangle$'le)

$\Rightarrow$ Define for each variable x

$$x = \begin{cases} 1 & \text{if } x \in \text{Ind-set} \\ 0 & \text{if } \overline{x} \in \text{Ind set} \\ \text{arbitrarily} & \text{otherwise} \end{cases}$$

No contradictions BECAUSE

x AND $\overline{x}$ never $\in$ Ind Set

EVERY CLAUSE is SATISFIED BECAUSE at least one literal is set to true

Since Ind set picks exactly one in every Δle.

≥ 1