CS 170 DIS 07

Released on 2018-10-15

1 Job Assignment

There are I people available to work J jobs. The value of person i working 1 day at job j is a_{ij} for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Each job is completed after the sum of the time of all workers spend on it add up to be 1 day, though partial completion still has value (i.e. person i working c portion of a day on job j is worth $a_{ij}c$). The problem is to find an optimal assignment of jobs for each person for one day.

(a) What variables should we optimize over? I.e. in the canonical linear programming definition, what is x?

Solution: An assignment x is a choice of numbers x_{ij} where x_{ij} is the portion of person i's time spent on job j.

(b) What are the constraints we need to consider? Hint: there are three major types.

Solution: First, no person i can work more than 1 day's worth of time.

$$\sum_{j=1}^{J} x_{ij} \leq 1 \qquad \text{for } i = 1, \ldots, I.$$

Second, no job j can be worked past completion:

$$\sum_{i=1}^{I} x_{ij} \leq 1 \qquad \text{for } j = 1, \ldots, J.$$

Third, we require positivity.

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \ldots, I, j = 1, \ldots, J.$$

(c) What is the maximization function we are seeking?

Solution: By person i working job j for x_{ij}, they contribute value $a_{ij}x_{ij}$. Therefore, the net value is

$$\sum_{i=1,j=1}^{I,J} a_{ij}x_{ij} = A \bullet x.$$

2 Understanding convex polytopes

So far in this class we have seen linear programming defined as

$$(\mathcal{P}) = \begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b. \end{cases}$$

Today, we explore the different properties of the region $\Omega = \{x : Ax \leq b\}$ – i.e. the region that our linear program maximizes over.

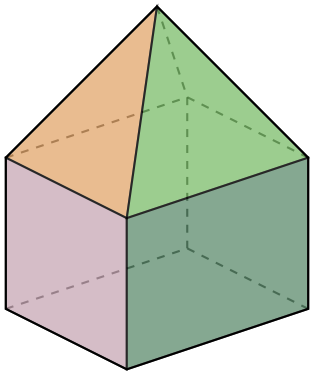


Figure 1: An example of a convex polytope. We can consider each face of the polytope as an affine inequality and then the polytope is all the points that satisfy each inequality. Notice that an affine inequality defines a half-plane and therefore is also the intersection of the half-planes.

(a) The first property that we will be interested in is *convexity*. We say that a space X is convex if for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)y \in X.$$

That is, the entire line segment $\overline{xy}$ is contained in X. Prove that Ω is indeed convex.

Solution: Let $x, y \in \Omega$. We need to show that

$$A(\lambda x + (1-\lambda)y)) \leq b.$$

We apply the only facts we know, namely $Ax \leq b$ and $Ay \leq b$.

$$\begin{aligned} A(\lambda x + (1-\lambda)y) &= \lambda Ax + (1-\lambda)Ay \\ &\leq \lambda b + (1-\lambda)b \\ &= b. \end{aligned}$$

(b) The second property that we will be interested in is showing that linear objective functions over convex polytopes achieve their maxima at the vertices. A vertex is any point $v \in \Omega$ such that v **cannot** be expressed as a point on the line $\overline{yz}$ for $v \neq y, v \neq z$, and $y, z \in \Omega$.

Prove the following statement: Let Ω be a convex space and f a linear function $f(x) = c^T x$. Show that the for a line $\overline{yz}$ for $y, z \in \Omega$ that $f(x)$ is maximized on the line at either y or z. I.e. show that

$$\max_{\lambda \in [0,1]} f(\lambda y + (1-\lambda)z)$$

achieves the maximum at either $\lambda = 0$ or $\lambda = 1$.

Solution: Assume without loss of generality that $f(y) \geq f(z)$. (Otherwise, swap their names). Then $c^T y \geq c^T z$. We now aim to show the maximum is achieved at $\lambda = 1$. Then,

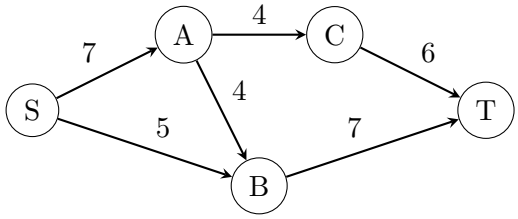
$$\begin{aligned} f(\lambda y + (1-\lambda)z) &= c^T(\lambda y + (1-\lambda)z) \\ &= \lambda c^T y + (1-\lambda)c^T z \\ &\leq \lambda c^T y + (1-\lambda)c^T y \\ &= f(y). \end{aligned}$$

(c) Now, prove that global maxima will be achieved at vertices. For simplicity, you can assume there is a unique global maximum. Hint: Use the definition of a vertex presented above. (*Side note:* This argument is the basis of the Simplex algorithm by Dantzig to solve linear programs.)

Solution: Assume, for contradiction, that the maximum was not achieved at a vertex and was instead achieved at a point x that was *not* a vertex. Then, there exists a line y, z containing x such that $x \neq y$ and $x \neq z$. But by the previous argument, the function achieves a maximum at either y or z. then, the maximum isn't unique. A contradiction.

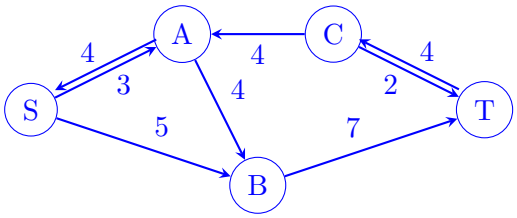
3 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

Solution:

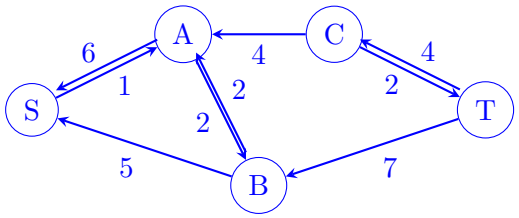


(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

Solution: A maximum flow of value 11 results from pushing:

- 4 units of flow through $S \to A \to C \to T$;
- 5 units of flow through $S \to B \to T$; and
- 2 units of flow through $S \to A \to B \to T$.

(There are other maximum flows of the same value, can you find them?) The resulting residual graph (with respect to the maximum flow above) is:



A minimum cut of value 11 is between $\{S, A, B\}$ and $\{C, T\}$ (with cross edges $A \to C$ and $B \to T$).

4 Verifying a max-flow

Suppose someone presents you with a solution to a max-flow problem on some network. Give a *linear* time algorithm to determine whether the solution does indeed give a maximum flow.

Solution: The max-flow algorithm has found the maximum flow when there is no $s - t$ path in the residual graph. Therefore, we just search for an $s - t$ path in the residual graph of the given flow to see if the given flow is maximal.

procedure CHECKFLOW(G, f)
Check that $\forall v \in V, v \neq s, t$, $\sum_{(u,v)\in E} f_{uv} = \sum_{(v,w)\in E} f_{vw}$
Compute G^f, the residual flow network of f.
Run BFS(G^f, s)
If BFS finds an s-t path, return false, otherwise return true.

Checking that f is a valid flow takes $O(|V| + |E|)$ time. Constructing G^f takes $O(|V| + |E|)$ time. Running BFS on G^f takes $O(|V| + |E|)$ time since G^f has $|V|$ vertices and $\leq 2|E|$ edges. Therefore, the algorithm is $O(|V| + |E|)$, which is linear.