CS 170 Dis 9

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1 Maximal Matching

Let $G = (V, E)$ be a (not necessarily bipartite) undirected graph. A *maximal matching*, M, is a matching in which no edge can be added while keeping it a matching. Show that the size of any maximal matching is at least half the size of a maximum matching M^*.

Solution: Assume for contradiction that there is a maximal matching M whose size is less than half the size of a maximal matching M^*. We will show that we can improve the matching on M by adding an edge from M^*, contradicting the claim that M is maximal. To see this note that for each $(u, v) \in M$, there are at most two edges in M^* incident on u or v (one for each vertex). Therefore, at most $2|M|$ edges in M^* are incident on some vertex appearing in M. Since $2|M| < |M^*|$, there is an edge $e \in M^*$ not incident on any vertex in M. So $M \cup \{e\}$ is a matching, and M is not maximal.

2 Bipartite Vertex Cover

A vertex cover of an undirected graph $G = (V, E)$ is a subset of the vertices which touches every edge. In other words, a subset $S \subset V$ such that for each edge $\{u, v\} \in E$, one or both of u, v are in S.

Show that the problem of finding the minimum vertex cover in a *bipartite* graph reduces to maximum flow. Prove that your reduction is correct.

Hint: use the max-flow min-cut theorem.

Solution: Let $L \cup R$ be the bipartition of the graph G. Construct a network G' by adding a dummy source node s, with edges going out to every vertex of L, and a dummy target node t, with edges coming in from every vertex of R. Direct the remaining original edges so that they go from L to R. Let the edges adjacent to s or t have capacity 1 and the original edges have infinite capacity.

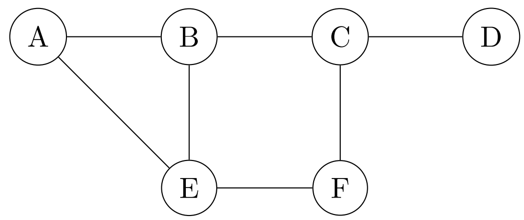
Consider any (s,t)-cut $(S, \bar{S})$ ($s \in S$) in this network which has size less than ∞. Let E_S be the set of edges crossing the cut from S to $\bar{S}$. Then for all $e \in E_S$, e is incident to either s or t (otherwise the cut contains an infinite capacity edge). Let C be the set of all vertices *except* s and t incident to edges in E. Then C is a vertex cover of G: if not, then there is some $\{u, v\} \in E$ for $u \in L, v \in R$ with $u, v \notin C$, so no edge on the path $s - u - v - t$ crosses the cut, a contradiction. Note $|C| = |E_S|$ which is the size of the cut $(S, \bar{S})$.

On the other hand, let C be a vertex cover of G. Let $S \subseteq V$ consist of the set of vertices in L which are *not* in C, the set of vertices in R which *are* in C, and s. Consider the set of edges E_S crossing the s-t cut from S to $\bar{S}$. First, suppose that E_S contains an infinite capacity edge $e = (u, v)$. Since $u \in S$, $u \notin C$, but then since $v \notin S$, $v \notin C$, and so C does not cover e. Hence E_S contains only edges with capacity 1. Moreover $|E_S| = |L \cap C| + |R \cap C| = |C|$, and so the size of the cut $(S, \bar{S})$ is $|C|$.

Let $(S, \bar{S})$ be a minimum cut in G'. Then C obtained as above is a minimum vertex cover of G: suppose not; then there is a smaller vertex cover C' of G, but then there is a smaller cut $(S', \bar{S}')$ in G'.

3 Reducing Vertex Cover to Set Cover

In the minimum vertex cover problem, we are given an undirected graph $G = (V, E)$ and asked to find the smallest set $U \subseteq V$ that "covers" the set of edges E. In other words, we want to find the smallest set U such that for each $(u, v) \in E$, either u or v is in U (U is not necessarily unique). For example, in the following graph, $\{A, E, C, D\}$ is a vertex cover, but not a minimum vertex cover. The minimum vertex covers are $\{B, E, C\}$ and $\{A, E, C\}$.



Recall the following definition of the minimum Set Cover problem: Given a set U of elements and a collection $S_1, \ldots, S_m$ of subsets of U, what is the smallest collection of these sets whose union equals U? So, for example, given $U := \{a, b, c, d\}$, $S_1 := \{a, b, c\}$, $S_2 := \{b, c\}$, and $S_3 := \{c, d\}$, a solution to the problem is the collection of S_1 and S_3.

Give an efficient reduction from the Minimum Vertex Cover Problem to the Minimum Set Cover Problem.

Solution: Let $G = (V, E)$ be an instance of the Minimum Vertex Cover Problem. Create an instance of the Minimum Set Cover Problem where $U = E$ and for each $u \in V$, the set S_u contains all edges adjacent to u. Let $C = \{S_{u_1}, S_{u_2}, \ldots, S_{u_k}\}$ be a set cover. Then our corresponding vertex cover will be $u_1, u_2, \ldots, u_k$. To see this is a vertex cover, take any $(u, v) \in E$. Since $(u, v) \in U$, there is some set S_{u_i} containing (u, v), so u_i equals u or v and (u, v) is covered in the vertex cover.

Now take any vertex cover $u_1, \ldots, u_k$. To see that $S_{u_1}, \ldots, S_{u_k}$ is a set cover, take any $(u, v) \in E$. By the definition of vertex cover, there is an i such that either $u = u_i$ or $v = u_i$. So $(u, v) \in S_{u_i}$, so $S_{u_1}, \ldots, S_{u_k}$ is a set cover.

Since every vertex cover has a corresponding set cover (and vice-versa) and minimizing set cover minimizes the corresponding vertex cover, the reduction holds.

4 Midterm Discussion

What did you find most challenging on the midterm? Are there any problems in particular you would like to discuss?