CS 170 DIS 10

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1 NP Basics

Assume A reduces to B in polynomial time. In each part you will be given a fact about one of the problems. What will you know about the other problem from each fact? *(You can answer each part in one sentence.)*

1. A is in **P**.
2. B is in **P**.
3. A is **NP**-hard.
4. B is **NP**-hard.

Solution: If A reduces to B, we know B can be used to solve A, which means B is at least as hard as A. As a result, in case 2 we can say that A is in **P**, and in case 3 we can say that B is **NP**-hard. In cases 1 and 4, we cannot conclude anything interesting.

2 Hitting Set

In the Hitting Set Problem, we are given a family of finite integer sets $\{S_1, S_2, \ldots, S_n\}$ and a budget b, and we wish to find an integer set H of size $\leq b$ which intersects every S_i, if such an H exists. In other words, we want $H \cap S_i \neq \emptyset$ for all i.

Show that the Hitting Set Problem is NP-complete.

Solution: This is a generalization of the Vertex-Cover Problem, which we saw last section. Given a graph G, consider each edge $e = (u, v)$ as a set containing the elements u and v. Then, finding a hitting set of size at most b in this particular family of sets is the same as finding a vertex cover of size at most b for the given graph.

We can see that the problem is in NP since we can quickly check that a potential hitting set covers all sets and has size less than b.

3 Reliable Network

Reliable Network is the following problem: We are given two $n \times n$ matrices (a cost matrix d_{ij} and a connectivity requirement matrix r_{ij}) and also a budget b. We want to find a graph $G = (\{1, ..., n\}, E)$ such that the total cost of all edges (i.e. $\sum_{(i,j) \in E} d_{ij}$) is at most b and there are exactly r_{ij} vertex-disjoint paths between any two distinct vertices i and j.

Show that Reliable Network is NP-Complete.

Solution: Reduction from Rudrata Cycle to Reliable Network. Given $G = (V, E)$, take $b = n$, $d_{ij} = 1\ \forall (i, j) \in E$ and $d_{ij} > 1$ otherwise; set $r_{ij} = 2\ \forall (i, j)$. Now we must find edges such that the sum of the weights of all the edges is n (so only n edges), and they are part of exactly one cycle (so there are two vertex-disjoint paths between each pair of vertices). This must contain all of the vertices since each pair satisfies this. This is exactly a Rudrata Cycle.

To prove Reliable Network is NP. Given a solution of Reliable Network (i.e. a graph G). We simply run max flow with $(s,t) = (i,j)$ for any i,j to verify the r_{ij} constraints.

4 Dominating Set

A dominating set of a graph $G = (V,E)$ is a subset D of V, such that every vertex not in D is a neighbor of at least one vertex in D.
Let the Minimum Dominating Set problem be the task of determining whether there is a dominating set of size $\leq k$.
Show that the Minimum Dominating Set problem is NP-Hard. You may assume for this question that all graphs are connected.

Solution:

We will reduce the Minimum Set Cover problem to the Minimum Dominating Set problem. Suppose (S,U) is an instance of set cover where U denotes the universe of distinct elements and S is a set of S_i, where each S_i is some subset of U.
We will construct a graph $G = (V,E)$ as follows.
For each element u in U construct a vertex. For each possible S_i construct a vertex. Connect each vertex S_i to all u in S_i.
Let's call "element vertices" vertices that correspond to elements in U, and "set vertices" vertices that correspond to some i.
Notice that if we were to run dominating set on the graph right now, we'd be able to cover all the element vertices with any valid set cover, but we'd have to pick every single set vertex in order to ensure that all set vertices are covered. To rectify this, connect every set vertex to every other set vertex, forming a clique. This ensures that we can cover all the set vertices by picking just one. This way we really only need to worry about covering the element vertices.
Suppose we have some Minimum set cover of size k. This will correspond to a dominating set of size k as well. For each set in our Minimum set cover, pick the corresponding set vertex. It follow directly from the construction of the graph and the definition of a set cover that all set and element vertices are covered.
Suppose we have some dominating set D of size k. We can find a set cover of size $\leq k$. To do this, we will construct a new dominating set D' that contains only set vertices. Include every set vertex in D in D'. For each vertex in our dominating set that is an element vertex, pick any random neighboring set vertex and add it to D'. Observe that $|D'| \leq |D|$.
Thus if there is a dominating set in G of size $\leq k$, there must be a set cover of size $\leq k$