CS 170 HW 8

Due on 2018-10-21, at 9:59 pm

1 (★) Study Group

List the names and SIDs of the members in your study group.

2 (★★) Jeweler

You are a jeweler who sells necklaces and rings. Each necklace takes 4 ounces of gold and 2 diamonds to produce, each ring takes 1 ounce of gold and 3 diamonds to produce. You have 80 ounces of gold and 90 diamonds. You make a profit of 60 dollars per necklace you sell and 30 dollars per ring you sell, and want to figure out how many necklaces and rings to produce to maximize your profits.

(a) Formulate this problem as a linear programming problem and find the solution (state the cost-function, linear constraints, and all vertices except for the origin).

(b) Suppose instead that the profit per necklace is C dollars and the profit per ring remains at 30 dollars. For each vertex you listed in the previous part, give the range of C values for which that vertex is the optimal solution.

3 (★★★★) Modeling: Tricks of the Trade

One of the most important problems in the field of *statistics* is the *linear regression problem*. Roughly speaking, this problem involves fitting a straight line to statistical data represented by points – $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ – on a graph. Denoting the line by $y = a + bx$, the objective is to choose the constants a and b to provide the "best" fit according to some criterion. The criterion usually used is the *method of least squares*, but there are other interesting criteria where linear programming can be used to solve for the optimal values of a and b. For each of the following criteria, formulate the linear programming model for this problem:

1. Minimize the sum of the absolute deviations of the data from the line; that is,

$$\min \sum_{i=1}^{n} |y_i - (a + bx_i)|$$

(*Hint:* Define a new variable $z_i = y_i - (a + bx_i)$. Notice that z_i can be either positive or negative. Any number, positive or negative, however, can be represented as the difference of two non-negative numbers. Also define as non-negative variables z_i^+ and z_i^- such that $z_i = z_i^+ - z_i^-$. How can we minimize $|z_i|$ by either minimizing or maximizing some function of z_i^+ and z_i^-?)

2. Minimize the maximum absolute deviation of the data from the line; that is,

$$\min \max_{i=1\ldots n} |y_i - (a + bx_i)|$$

(*Hint:* You'll need to start by using the same trick as above. Then consider how we can turn our objective function into a single minimization or maximization.)

4 (★★) Repairing a Flow

In a particular network $G = (V, E)$ whose edges have integer capacities c_e, we have already found a maximum flow f from node s to node t where f_e is an integer for every edge. However, we now find out that one of the capacity values we used was wrong: for edge (u, v) we used c_{uv} whereas it should have been $c_{uv} - 1$. This is unfortunate because the flow f uses that particular edge at full capacity: $f_{uv} = c_{uv}$. We could redo the flow computation from scratch, but there's a faster way.

Describe an algorithm to repair the max-flow in $O(|V| + |E|)$ time. Also give a proof of correctness and runtime justification.

5 (★★★) Generalized Max Flow

Consider the following generalization of the maximum flow problem.

You are given a directed network $G = (V, E)$ where edge e has capacity c_e. Instead of a single (s, t) pair, you are given multiple pairs $(s_1, t_1), ..., (s_k, t_k)$, where the s_i are sources of G and t_i are sinks of G. You are also given k (positive) demands $d_1, ..., d_k$. The goal is to find k flows $f^{(1)}, ..., f^{(k)}$ with the following properties:

(a) $f^{(i)}$ is a valid flow from s_i to t_i.

(b) For each edge e, the total flow $f_e^{(1)} + f_e^{(2)} + ... + f_e^{(k)}$ does not exceed the capacity c_e.

(c) The size of each flow $f^{(i)}$ is at least the demand d_i.

(d) The size of the *total* flow (the sum of the flows) is as large as possible.

Write a linear problem using the variables $f_e^{(i)}$ whose optimal solution is exactly the solution to this problem. For each constraint as well as the objective in your linear program briefly explain why it is correct. (Note: Since linear programs can be solved in polynomial time, this implies a polynomial-time algorithm for the problem)

6 (★★★★) Reductions Among Flows

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a directed graph G and the additional variant constraints, show how to construct a directed graph G' such that

(1) If F is a flow in G satisfying the additional constraints, there is a flow F' in G' of the same size,

(2) If F' is a flow in G', then there is a flow F in G satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph G'.

(a) **Max-Flow with Vertex Capacities:** In addition to edge capacities, every vertex $v \in G$ has a capacity c_v, and the flow must satisfy $\forall v : \sum_{u:(u,v)\in E} f_{uv} \leq c_v$.

(b) **Max-Flow with Multiple Sources:** There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

7 (★★) Provably Optimal

For the linear program

$$\begin{aligned} \max\ & x_1 - 2x_3 \\ & x_1 - x_2 \leq 1 \\ & 2x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You do not have to solve for the optimum of the dual.

(*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

8 (★★★) Bimetallism

There is a blacksmith who can produce n different alloys, where alloy i sells for p_i dollars per unit. One unit of alloy i takes g_i grams of gold and s_i grams of silver to produce. The blacksmith has a total of G grams of gold and S grams of silver to work with, and can produce as many units of each type of alloy as they want within the material constraints. The blacksmith is allowed to produce and sell a non-integer number of units of each alloy.

1. Formulate the linear program to maximize the revenue of the blacksmith, and explain your decision variables, objective function, and constraints.

2. Formulate the dual of the linear program from part (a), and explain your decision variables, objective function, and constraints. The explanations provide economic intuition behind the dual. We will only be grading the dual formulation.
   Hint: Formulate the dual first, then think about it from the perspective of the blacksmith when negotiating prices for buying G grams of gold and S grams of silver if they had already signed a contract for the prices for the output alloys p_i. Think about the breakeven point, from which the blacksmith's operations begin to become profitable for at least one alloy.