CS 170 HW 9

Due on 2018-10-28, at 9:59 pm

1 (★) Study Group

List the names and SIDs of the members in your study group.

2 (★★★) A cohort of spies

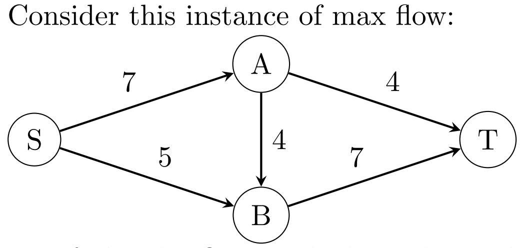
A cohort of k spies resident in a certain country needs escape routes in case of emergency. They will be travelling using the railway system which we can think of as a directed graph $G = (V, E)$ with V being the cities. Each spy i has a starting point $s_i \in V$ and needs to reach the consulate of a friendly nation; these consulates are in a known set of cities $T \subseteq V$. In order to move undeteced, the spies agree that at most c of them should ever pass through any one city. Our goal is to find a set of paths for each of the spies (or detect that the requirements cannot be met).

Model this problem as a flow network. Specify the vertices, edges and capacities, and show that a maximum flow in your network can be transformed into an optimal solution for the original problem. You do not need to explain how to solve the max-flow instance itself.

3 (★★★) Max Flow, Min Cut, and Duality

In this exercise, we will demonstrate that LP duality can be used to show the max-flow min-cut theorem.

Consider this instance of max flow:



Let f_1 be the flow pushed on the path $\{S, A, T\}$, f_2 be the flow pushed on the path $\{S, A, B, T\}$, and f_3 be the flow pushed on the path $\{S, B, T\}$. The following is an LP for max flow in terms of the variables f_1, f_2, f_3:

$$
\begin{aligned}
\max \quad & f_1 + f_2 + f_3 \\
& f_1 + f_2 \leq 7 && \text{(Constraint for } (S, A)\text{)} \\
& f_3 \leq 5 && \text{(Constraint for } (S, B)\text{)} \\
& f_1 \leq 4 && \text{(Constraint for } (A, T)\text{)} \\
& f_2 \leq 4 && \text{(Constraint for } (A, B)\text{)} \\
& f_2 + f_3 \leq 7 && \text{(Constraint for } (B, T)\text{)} \\
& f_1, f_2, f_3 \geq 0
\end{aligned}
$$

The objective is to maximize the flow being pushed, with the constraint that for every edge, we can't push more flow through that edge than its capacity allows.

(a) Find the dual of this linear program, where the variables in the dual are x_e for every edge e in the graph.

(b) Consider any cut in the graph. Show that setting $x_e = 1$ for every edge crossing this cut and $x_e = 0$ for every edge not crossing this cut gives a feasible solution to the dual program.

(c) Based on your answer to the previous part, what problem is being modelled by the dual program? By LP duality, what can you argue about this problem and the max flow problem?

4 (★★) Zero-Sum Battle

Two Pokemon trainers are about to engage in battle! Each trainer has 3 Pokemon, each of a single, unique type. They each must choose which Pokemon to send out first. Of course each trainer's advantage in battle depends not only on their own Pokemon, but on which Pokemon their opponent sends out.

The table below indicates the competitive advantage (payoff) Trainer A would gain (and Trainer B would lose). For example, if Trainer B chooses the fire Pokemon and Trainer A chooses the rock Pokemon, Trainer A would have payoff 2.

		Trainer B: ice	grass	fire
Trainer A:	dragon	-10	3	3
	steel	4	-1	-3
	rock	6	-9	2

Feel free to use an online LP solver to solve your LPs in this problem.
Here is an example of a Python LP Solver and its Tutorial.

1. Write an LP to find the optimal strategy for Trainer A. What is the optimal strategy and expected payoff?

2. Now do the same for Trainer B. What is the optimal strategy and expected payoff?

5 (★★★★) Minimum Spanning Trees

Consider the minimum spanning tree problem, where we are given an undirected graph G with edge weights $w_{u,v}$ for every pair of vertices u, v.

An *integer* linear program that solves the minimum spanning tree problem is as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{(u,v)\in E} w_{u,v} x_{u,v} \\ \text{subject to} \quad & \sum_{\{u,v\}\in E: u\in S, v\in V\setminus S} x_{u,v} \geq 1 \quad \text{for all } S \subseteq V \text{ with } 0 < |S| < |V| \\ & \sum_{\{u,v\}\in E} x_{u,v} \leq |V| - 1 \\ & x_{u,v} \in \{0,1\}, \quad \forall (u,v) \in E \end{aligned}$$

(a) Show how to obtain a minimum spanning tree T of G from an optimum solution of the ILP, and prove that T is indeed an MST. Why do we need the constraint $x_{u,v} \in \{0,1\}$?

(b) How many constraints does the program have?

(c) Suppose that we *replaced* the binary constraint on each of the decision variables $x_{u,v}$ with the pair of constraints:

$$0 \leq x_{u,v} \leq 1, \quad \forall (u,v) \in E$$

How does this affect the optimum value of the program? Give an example of a graph where the optimum value of the relaxed linear program differs from the optimum value of the integer linear program.