

CS170 - Lecture 5

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UC Berkeley

Recall: Multiplying polynomials, coefficient/value representation

Coefficient representation:

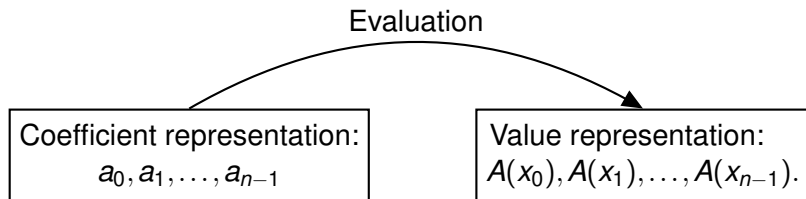
$$a_0, a_1, \dots, a_{n-1}$$

Value representation:

$$A(x_0), A(x_1), \dots, A(x_{n-1}).$$

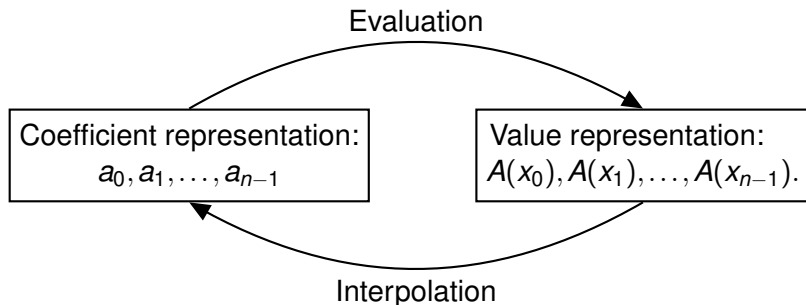
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Evaluation: $O(n \log n)$ if choose $1, \omega, \omega^2, \dots, \omega^{n-1}$.



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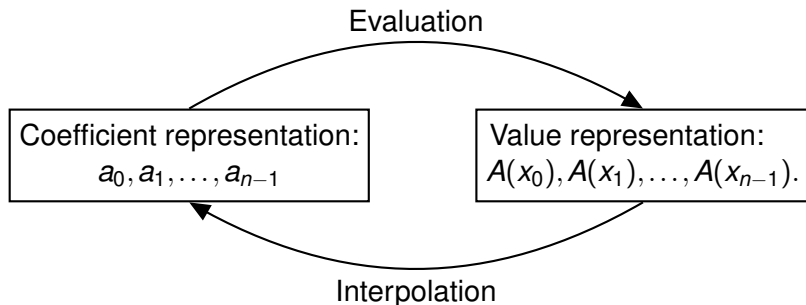
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Interpolation: From points $A(x_0), \dots, A(x_{n-1})$ to coefficients..

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Interpolation: From points $A(x_0), \dots, A(x_{n-1})$ to coefficients..
We will see this today!

Polynomial Evaluation and Matrices

Evaluation: Compute $A(\cdot)$ from a_i 's:

$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ & & \vdots & \cdots & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Interpolation (going back to coefficient matrix).

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Compute inverse of matrix above.

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Multiply.

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Interpolation (going back to coefficient matrix).

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This sounds expensive!!

Also, computing inverse not even easy.

Using roots of unity

FFT: ω is complex n th root of unity

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Compute inverse of $M_n(\omega)$?

Geometry and FFT.

Rows are orthogonal.

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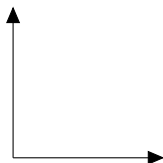
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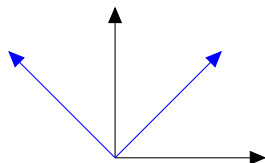


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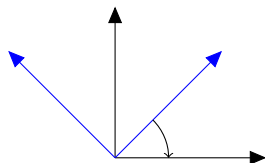


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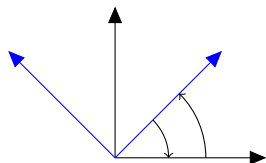


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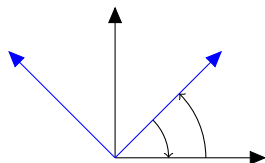
Reverse Rotation is inverse operation.

Geometry and FFT.

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Rigid Rotation (and scaling.)!



Reverse Rotation is inverse operation.

Scaling: for rotation, axis should have length 1, FFT length n .

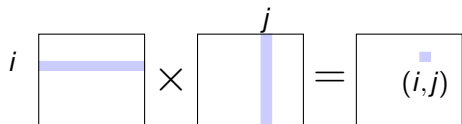
Algebraically.

Inversion formula: $(M_n(\omega))^{-1} = \frac{1}{n} M_n(\omega^{-1})$.

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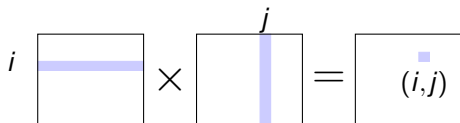
$C =$
 $M_n(\omega) \times M_n(\omega^{-1})?$



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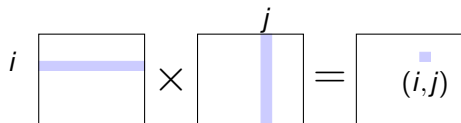
Recall: $\omega = e^{2\pi/n}$.

$$c_{ij} = \sum_k \omega^{ik} \omega^{-kj}$$

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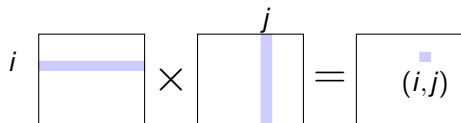
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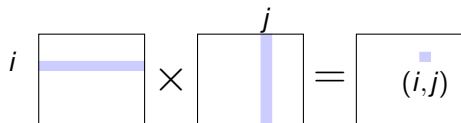
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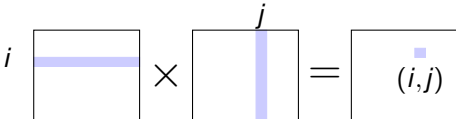


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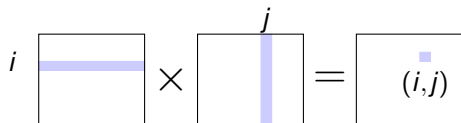
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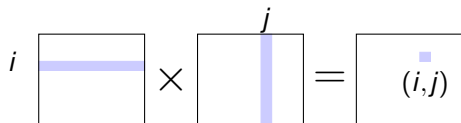
Case $i = j$: $r = \omega^0 = 1$

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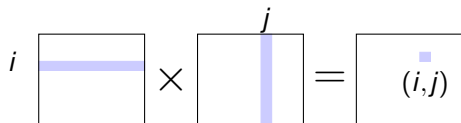
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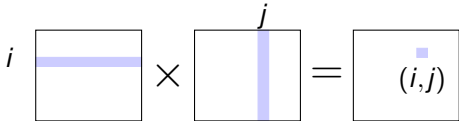
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Inversion formula: $(M_n(\omega))^{-1} = \frac{1}{n} M_n(\omega^{-1})$.

$$C = M_n(\omega) \times M_n(\omega^{-1})?$$


The diagram shows three boxes. The first box is a square with a horizontal blue bar across its middle, labeled with the letter i to its left. This is multiplied by a second square box with a vertical blue bar down its middle, labeled with the letter j above it. The result is a third square box containing a small blue square at the intersection of the horizontal and vertical bars, with the label (i, j) below it.

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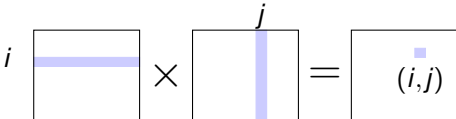
Case $i = j$: $r = \omega^0 = 1$ and $c_{ij} = n$.

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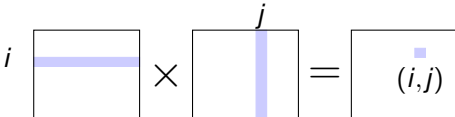
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The diagram shows three square boxes representing matrices. The first box has a horizontal blue bar across its middle, with the letter 'i' to its left. The second box has a vertical blue bar down its middle, with the letter 'j' above it. A multiplication symbol '×' is between the first and second boxes. An equals sign '=' is between the second and third boxes. The third box contains a small blue square at the intersection of the horizontal and vertical lines from the first two boxes, with the text '(i,j)' below it.

Recall: $\omega = e^{2\pi/n}$.

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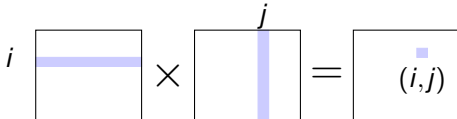
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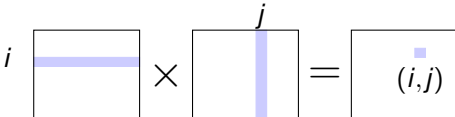
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Recall: $\omega = e^{2\pi/n}$.

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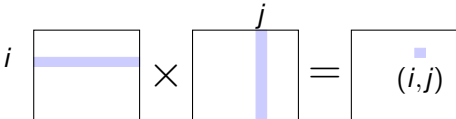
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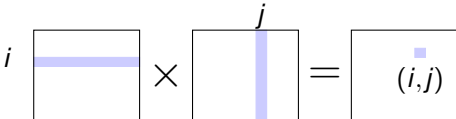
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For C – diagonals are n and the off-diagonals are 0.

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The diagram shows three square matrices. The first matrix has a horizontal blue bar representing row i . The second matrix has a vertical blue bar representing column j . An 'X' symbol is between them. An equals sign follows, leading to a third matrix with a small blue square at the intersection of row i and column j , labeled (i,j) .

Recall: $\omega = e^{2\pi/n}$.

$$c_{ij} = \sum_k \omega^{ik} \omega^{-kj} = \sum_k \omega^{(ik-kj)} = \sum_k \omega^{k(i-j)} = \sum_k r^k, \quad r = \omega^{(i-j)}$$

Case $i = j$: $r = \omega^0 = 1$ and $c_{ij} = n$.

Case $i \neq j$:

$$c_{ij} = 1 + r + r^2 + \dots + r^{n-1} = \frac{1-r^n}{1-r}$$

$$r^n = (\omega^{(i-j)})^n = (\omega^n)^{(i-j)} = 1^{(i-j)} \implies c_{ij} = 0.$$

For C – diagonals are n and the off-diagonals are 0.

Divide by n get identity!

Algebraically.

Inversion formula: $(M_n(\omega))^{-1} = \frac{1}{n} M_n(\omega^{-1})$.

$$C = M_n(\omega) \times M_n(\omega^{-1})?$$

The diagram shows three boxes. The first box is a square with a horizontal blue line across its middle, labeled with an italic i to its left. The second box is a square with a vertical blue line down its middle, labeled with an italic j above it. A large \times symbol is between the first and second boxes. An equals sign follows, leading to a third box. This third box is a square with a small blue square at its top-right corner, labeled with (i, j) below it.

Recall: $\omega = e^{2\pi/n}$.

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Computing inverse.

FFT works with points with basic root of unity: ω or ω^{-1}

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 $1, \omega^{-1}, \omega^{-2}, \dots, \omega^{-(n-1)}$.

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ω^{-1} is a primitive n th root of unity!

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FFT works with points with basic root of unity: ω or ω^{-1}

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Evaluation: FFT(a, ω).

Computing inverse.

FFT works with points with basic root of unity: ω or ω^{-1}
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ω^{-1} is a primitive n th root of unity!

Evaluation: $\text{FFT}(\mathbf{a}, \omega)$.

Interpolation: $\frac{1}{n} \text{FFT}(\mathbf{a}, \omega^{-1})$.

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$\implies O(n \log n)$ time for multiplying degree n polynomials.

Multiplying polynomials?

Coefficient representation:

a_0, a_1, \dots, a_{n-1}

+ is $O(n)$, * is $O(n^2)$ or $O(n^{\log_2 3})$

Value representation:

$A(x_0), A(x_1), \dots, A(x_{n-1})$.

+ is $O(n)$, * is $O(n)$

Multiplying polynomials?

Evaluation: $O(n \log n)$ if choose $1, \omega, \omega^2, \dots, \omega^{n-1}$.

Evaluation with $FFT(\omega)$ $O(n \log n)$



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a_0, a_1, \dots, a_{n-1}

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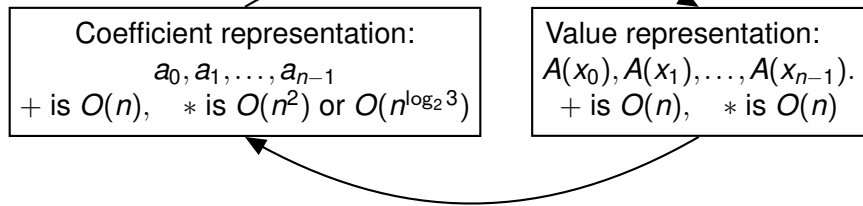
$A(x_0), A(x_1), \dots, A(x_{n-1})$.

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Multiplying polynomials?

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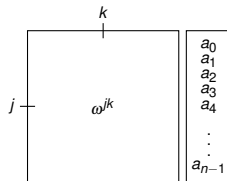
Evaluation with $FFT(\omega)$ $O(n \log n)$



Interpolation with $FFT(\omega^{-1})$ $O(n \log n)$

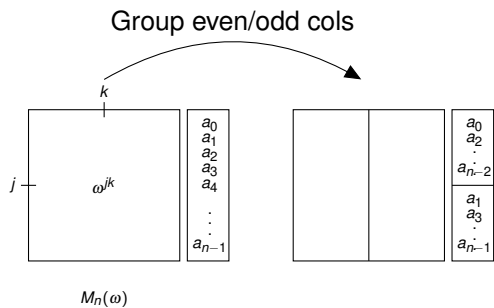
Interpolation: From points $A(x_0), \dots, A(x_{n-1})$ to "function".

FFT: a closer look.

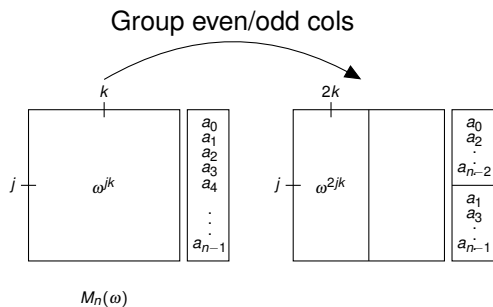


$M_n(\omega)$

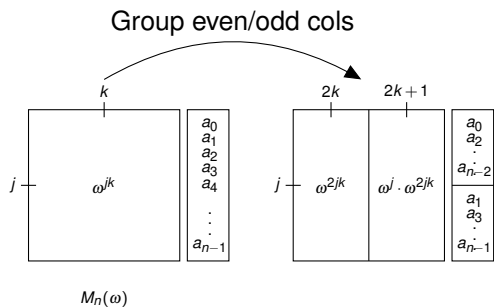
FFT: a closer look.



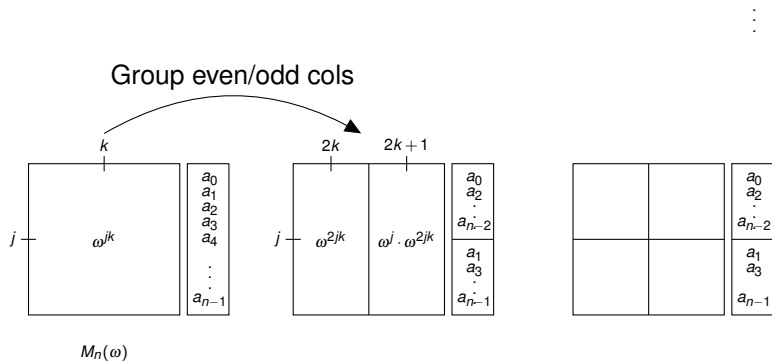
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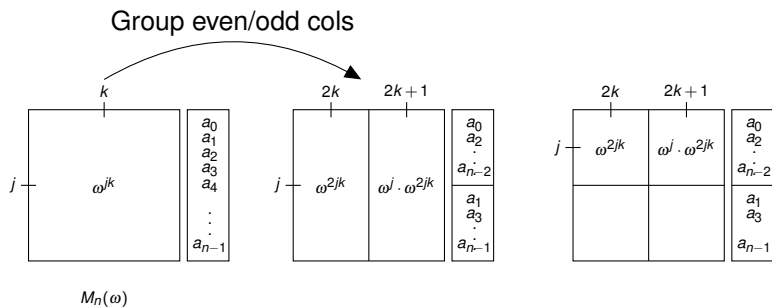
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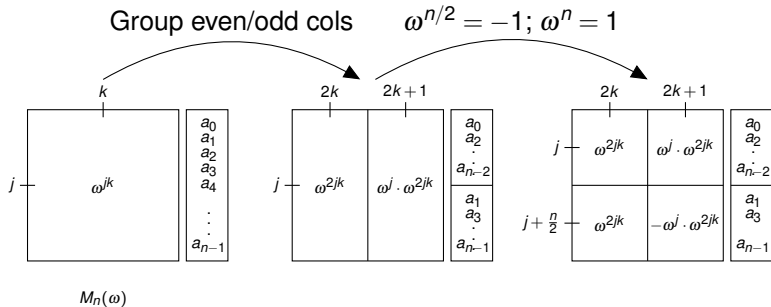
FFT: a closer look.



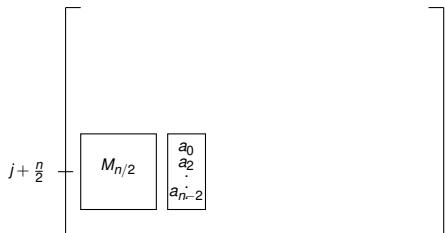
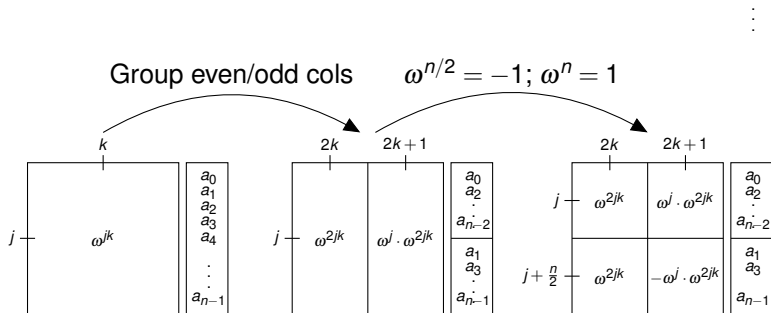
⋮

FFT: a closer look.

⋮

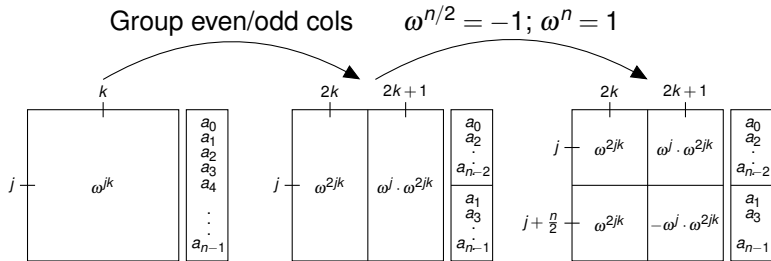


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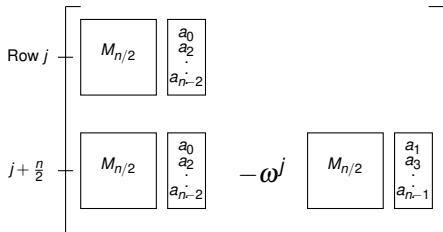


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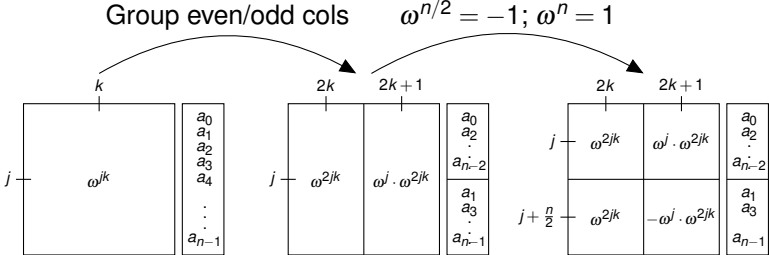


$M_n(\omega)$



FFT: a closer look.

⋮



$M_n(\omega)$

$$\begin{array}{l}
 \text{Row } j \\
 \\
 j + \frac{n}{2}
 \end{array}
 \left[
 \begin{array}{c}
 \begin{array}{|c|} \hline M_{n/2} \\ \hline \end{array}
 \begin{array}{|c|} \hline a_0 \\ a_2 \\ \vdots \\ a_{n-2} \\ \hline \end{array}
 + \omega^j
 \begin{array}{|c|} \hline M_{n/2} \\ \hline \end{array}
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Definitive Algorithm: FFT

FFT: " $M(\omega)a$ "

Definitive Algorithm: FFT

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Idea:

Definitive Algorithm: FFT

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Idea:

“ $M(\omega)a$ ” computed from...

Definitive Algorithm: FFT

FFT: " $M(\omega)a$ "

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if $\omega = 1$ return a

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for $j = 0$ to $n/2 - 1$:

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for $j = 0$ to $n/2 - 1$:

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return $(r_0, r_1, \dots, r_{n-1})$

Runtime: $T(n) = 2T(n/2) + O(n)$

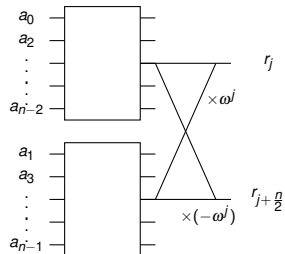
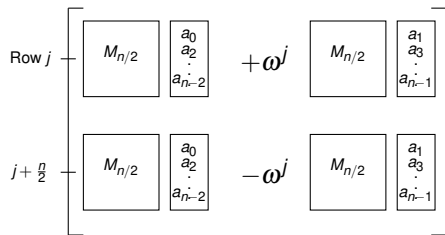
Unfolding FFT.

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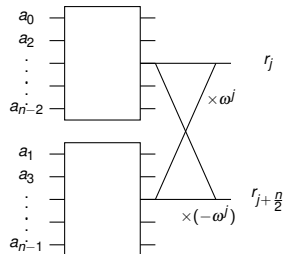
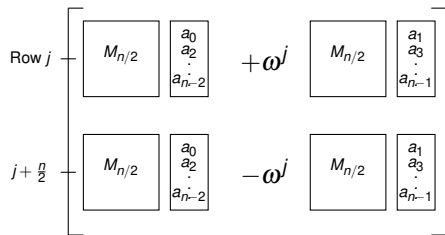
$$\begin{array}{l} \text{Row } j \\ j + \frac{n}{2} \end{array} \left[\begin{array}{cc} \boxed{M_{n/2}} & \begin{array}{c} a_0 \\ a_2 \\ \vdots \\ a_{n-2} \end{array} \\ \boxed{M_{n/2}} & \begin{array}{c} a_0 \\ a_2 \\ \vdots \\ a_{n-2} \end{array} \end{array} \right] \begin{array}{c} + \omega^j \\ - \omega^j \end{array} \left[\begin{array}{cc} \boxed{M_{n/2}} & \begin{array}{c} a_1 \\ a_3 \\ \vdots \\ a_{n-1} \end{array} \\ \boxed{M_{n/2}} & \begin{array}{c} a_1 \\ a_3 \\ \vdots \\ a_{n-1} \end{array} \end{array} \right]$$

Butterfly switches!

Unfolding FFT.

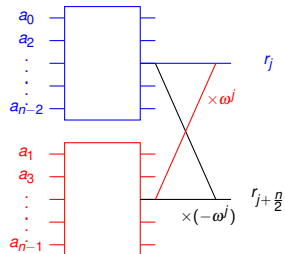
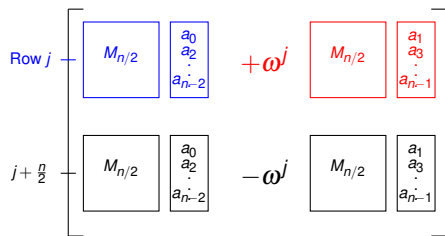


Unfolding FFT.



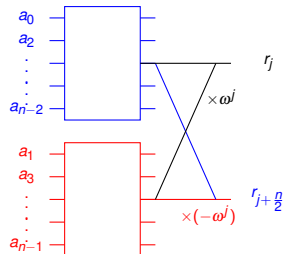
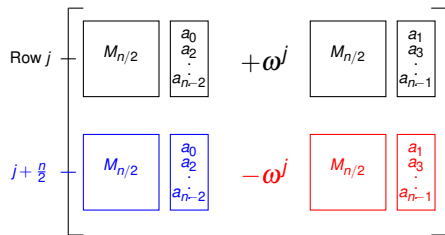
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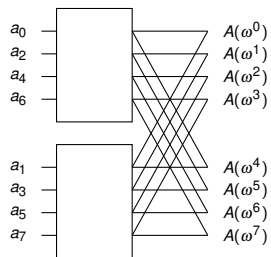
Butterfly switches!

Unfolding FFT.

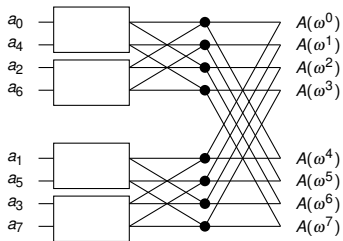
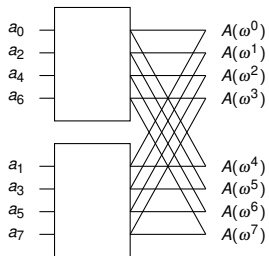


Butterfly switches!

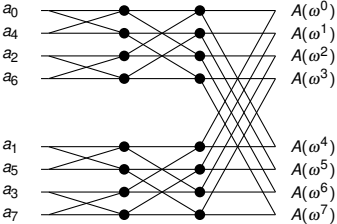
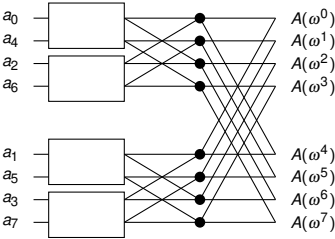
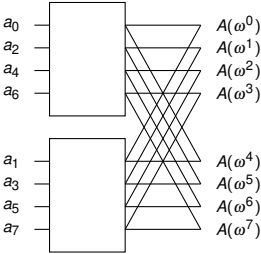
Expanding FFT...



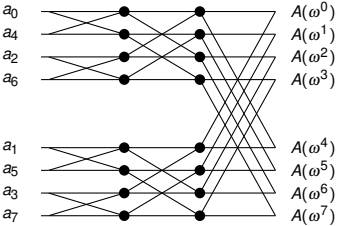
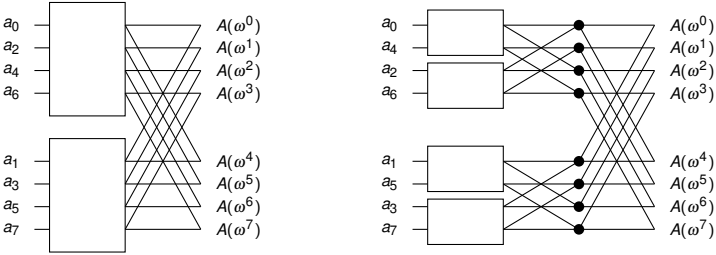
Expanding FFT...



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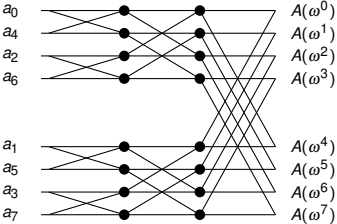
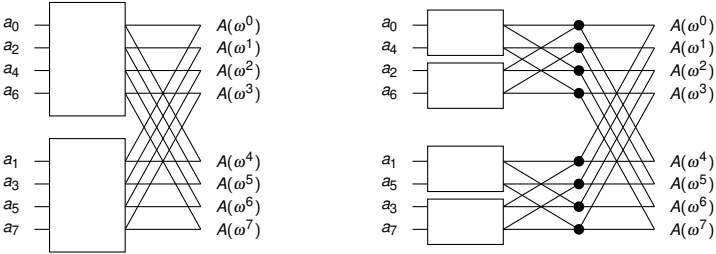


Expanding FFT...



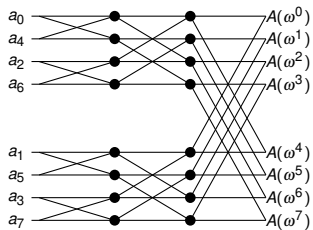
Edges from lower half of FFT have multipliers!

Expanding FFT...

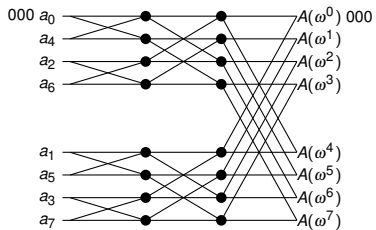


Edges from lower half of FFT have multipliers!

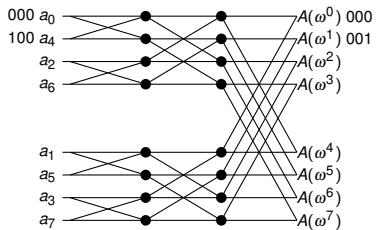
Order on Left



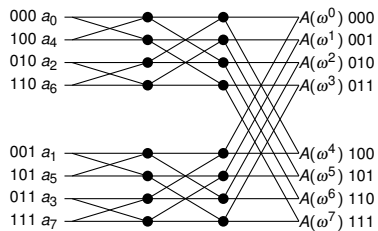
Order on Left



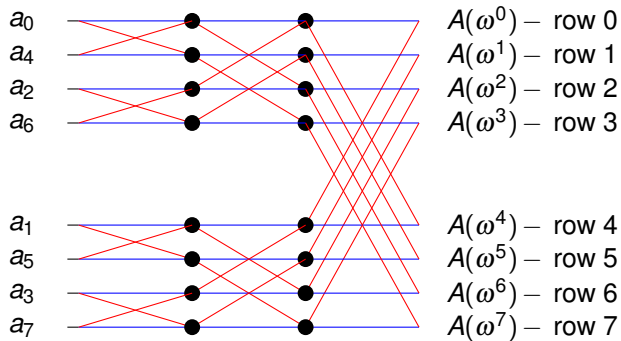
Order on Left



Order on Left

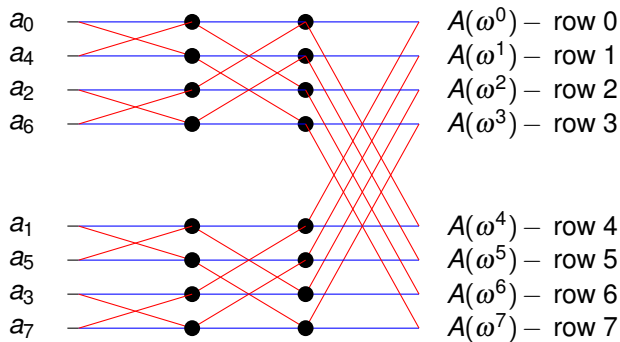


FFT Network.



$\log N$ - levels.

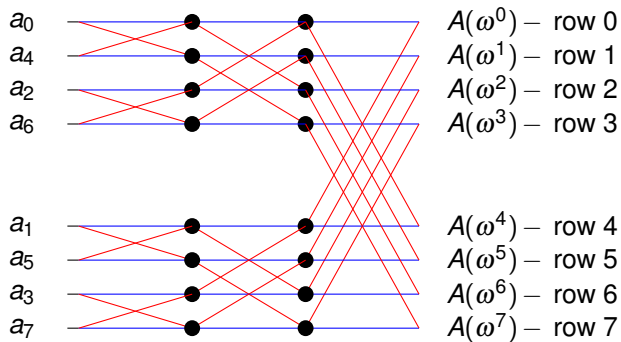
FFT Network.



$\log N$ - levels.

N - rows.

FFT Network.

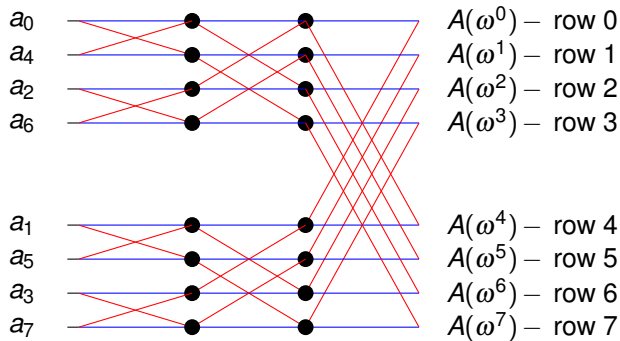


$\log N$ - levels.

N - rows.

In level i :

FFT Network.



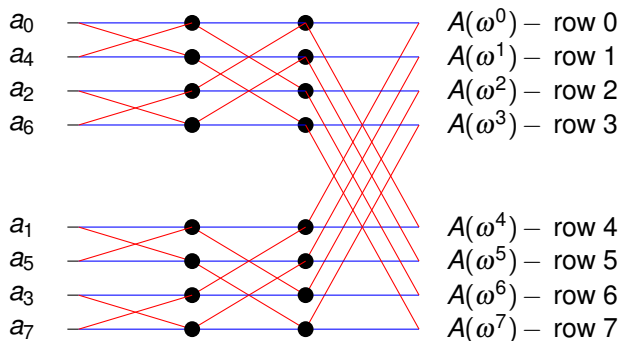
$\log N$ - levels.

N - rows.

In level i :

Row r node is connected to row r node in level $i + 1$.

FFT Network.



$\log N$ - levels.

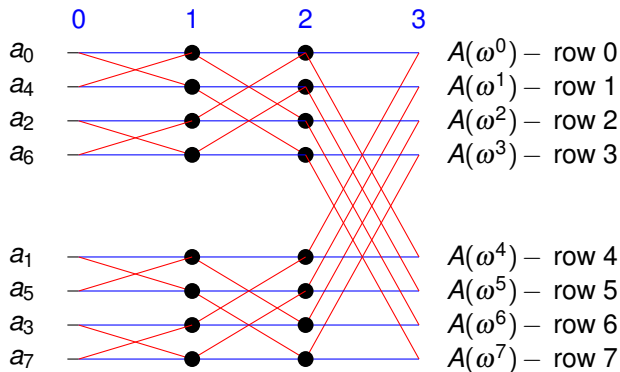
N - rows.

In level i :

Row r node is connected to row r node in level $i+1$.

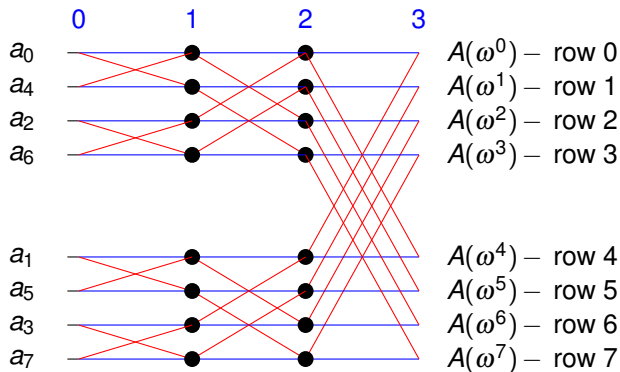
Row r node connected to row $r \pm 2^i$ node in level $i+1$

FFT Network.



Row r node connected to row $r \pm 2^i$ node in level $i + 1$

FFT Network.

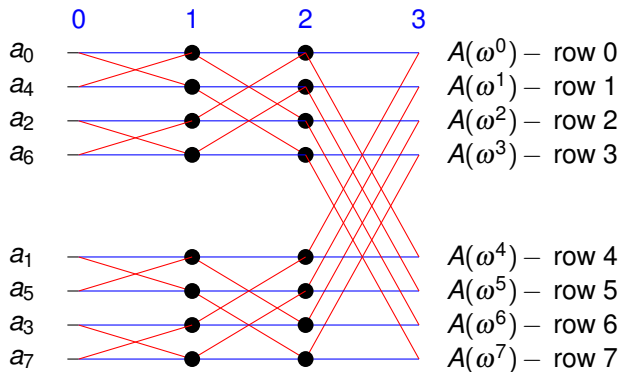


Row r node connected to row $r \pm 2^i$ node in level $i + 1$

When is it $r + 2^i$?

- (A) When $\lfloor r/2^i \rfloor$ is odd.
- (B) When $\lfloor r/2^i \rfloor$ is even.

FFT Network.

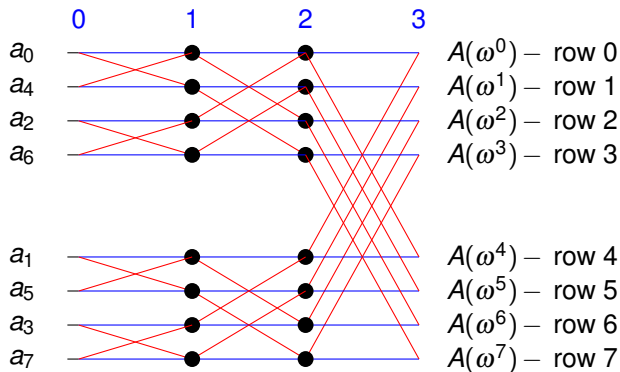


Row r node connected to row $r \pm 2^i$ node in level $i + 1$

When is it $r + 2^i$?

- (A) When $\lfloor r/2^i \rfloor$ is odd.
- (B) When $\lfloor r/2^i \rfloor$ is even.
- (B).

FFT Network.

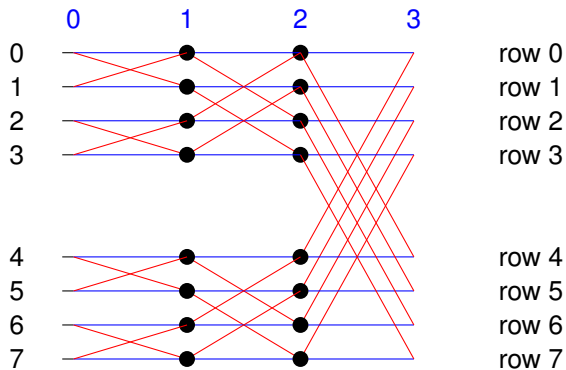


Row r node connected to row $r \pm 2^i$ node in level $i + 1$

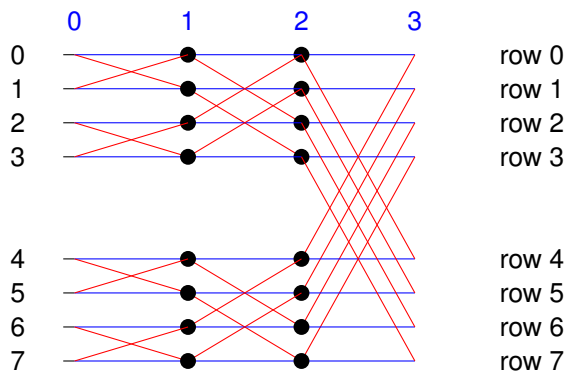
When is it $r + 2^i$?

- (A) When $\lfloor r/2^i \rfloor$ is odd.
- (B) When $\lfloor r/2^i \rfloor$ is even.
- (B). Red edges flip bit!

Unique Paths.

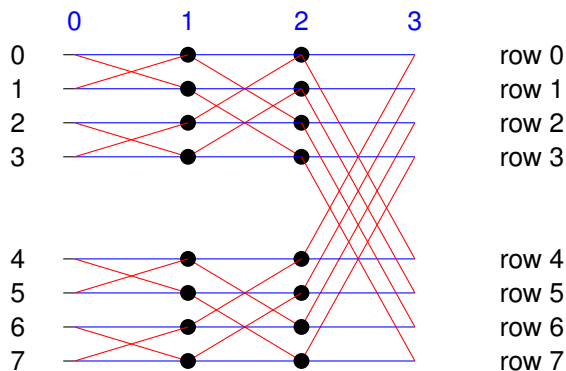


Unique Paths.



Route from input $i = 101$ to output $j = 000$?

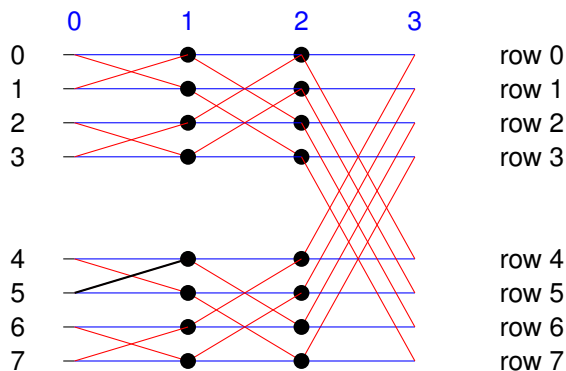
Unique Paths.



Route from input $i = 101$ to output $j = 000$?

Flip first bit.

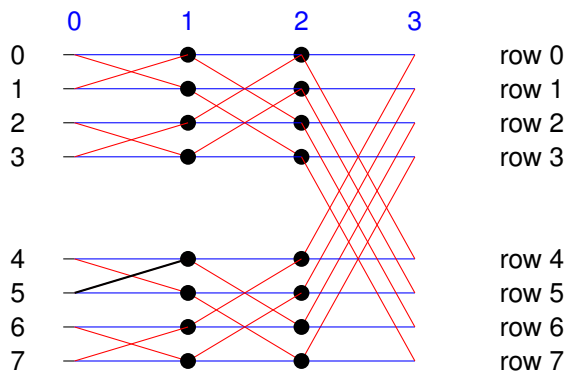
Unique Paths.



Route from input $i = 101$ to output $j = 000$?

Flip first bit. Red (cross) edge.

Unique Paths.

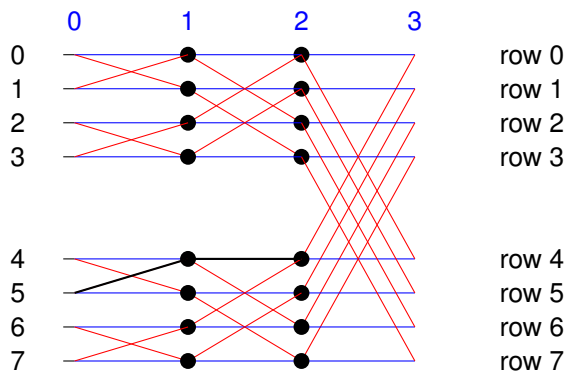


Route from input $i = 101$ to output $j = 000$?

Flip first bit. Red (cross) edge.

Keep second bit.

Unique Paths.

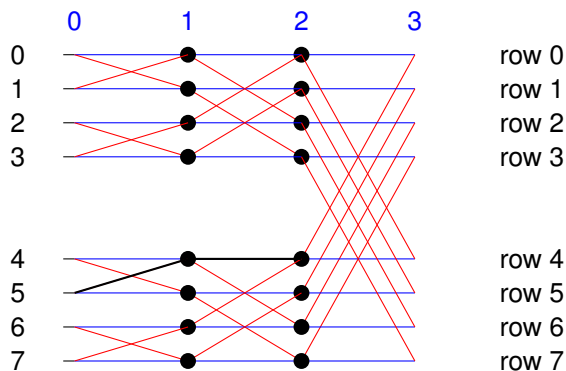


Route from input $i = 101$ to output $j = 000$?

Flip first bit. Red (cross) edge.

Keep second bit. Blue (straight) edge.

Unique Paths.



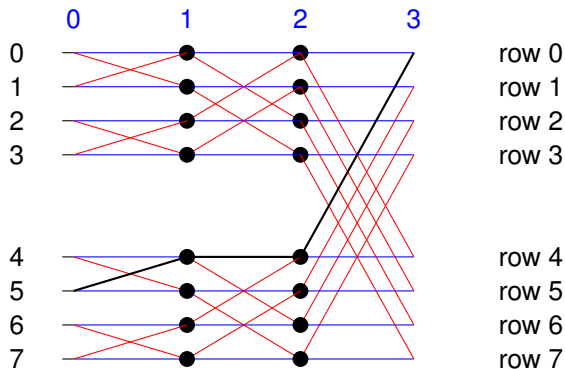
Route from input $i = 101$ to output $j = 000$?

Flip first bit. Red (cross) edge.

Keep second bit. Blue (straight) edge.

Flip third bit.

Unique Paths.



Route from input $i = 101$ to output $j = 000$?

Flip first bit. Red (cross) edge.

Keep second bit. Blue (straight) edge.

Flip third bit. Red (cross edge).

Summary.

Definitive FFT algorithm and code.

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x),$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

$$B(x) = \sum_{i=0}^d b_i x^i = B_L(x) + x^{d/2} B_H(x),$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

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$$B(x) = \sum_{i=0}^d b_i x^i = B_L(x) + x^{d/2} B_H(x), B_L(x) := \sum_{i=0}^{d/2} b_i x^i, B_H(x) = \sum_{i=1}^{d/2} b_{i+d/2} x^i$$

The product $A(x)B(x)$ is

$$A_L(x)B_L(x) +$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

$$B(x) = \sum_{i=0}^d b_i x^i = B_L(x) + x^{d/2} B_H(x), B_L(x) := \sum_{i=0}^{d/2} b_i x^i, B_H(x) = \sum_{i=1}^{d/2} b_{i+d/2} x^i$$

The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x))$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

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The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x)) + x^d A_H(x)B_H(x)$$

Compute ...

$$A_L(x)B_L(x),$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

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The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x)) + x^d A_H(x)B_H(x)$$

Compute ...

$$A_L(x)B_L(x), \quad A_H(x)B_H(x),$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

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The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x)) + x^d A_H(x)B_H(x)$$

Compute ...

$$A_L(x)B_L(x), \quad A_H(x)B_H(x), \quad (A_L(x) + A_H(x))(B_L(x) + B_H(x))$$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

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The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x)) + x^d A_H(x)B_H(x)$$

Compute ...

$$A_L(x)B_L(x), \quad A_H(x)B_H(x), \quad (A_L(x) + A_H(x))(B_L(x) + B_H(x))$$

and recurse

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

$$B(x) = \sum_{i=0}^d b_i x^i = B_L(x) + x^{d/2} B_H(x), B_L(x) := \sum_{i=0}^{d/2} b_i x^i, B_H(x) = \sum_{i=1}^{d/2} b_{i+d/2} x^i$$

The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x)) + x^d A_H(x)B_H(x)$$

Compute ...

$$A_L(x)B_L(x), \quad A_H(x)B_H(x), \quad (A_L(x) + A_H(x))(B_L(x) + B_H(x))$$

and recurse

Time is $O(d^{\log_2 3})$

$d^{\log_2 3}$ Polynomial Multiplication - Divide and Conquer.

$$A(x) = \sum_{i=0}^d a_i x^i = A_L(x) + x^{d/2} A_H(x), A_L(x) := \sum_{i=0}^{d/2} a_i x^i, A_H(x) = \sum_{i=1}^{d/2} a_{i+d/2} x^i$$

$$B(x) = \sum_{i=0}^d b_i x^i = B_L(x) + x^{d/2} B_H(x), B_L(x) := \sum_{i=0}^{d/2} b_i x^i, B_H(x) = \sum_{i=1}^{d/2} b_{i+d/2} x^i$$

The product $A(x)B(x)$ is

$$A_L(x)B_L(x) + x^{d/2}(A_L(x)B_H(x) + A_H(x)B_L(x)) + x^d A_H(x)B_H(x)$$

Compute ...

$$A_L(x)B_L(x), \quad A_H(x)B_H(x), \quad (A_L(x) + A_H(x))(B_L(x) + B_H(x))$$

and recurse

Time is $O(d^{\log_2 3})$

FFT does better. (But this is useful to see)