CS170 - Lecture 6
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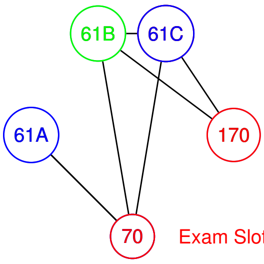
Today

1. Graphs
2. Depth First Search
3. Reachability

Scheduling: coloring.



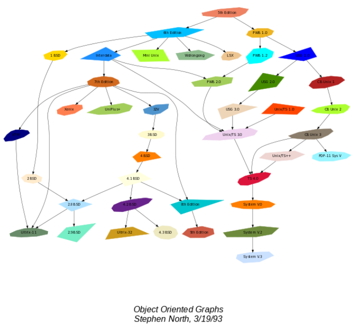
Exam Slot 1.

Exam Slot 2.

Exam Slot 3.

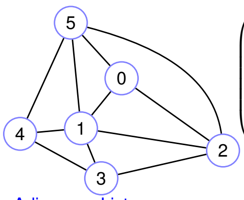
Directed acyclic graphs.

Heritage of Unix.



From http://www.graphviz.org/content/crazy.

Graph $G = (V, E)$.



Matrix Representation.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

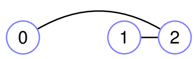
$V = \{0, 1, 2, 3, 4, 5\}$
$E = \{(0,1), (0,2), (0,5), (1,3) \ldots\}$

Adjacency List

0 : 1, 2, 5
1 : 0, 2, 3, 4, 5
2 : 0, 1, 3, 5
3 : 1, 2, 4
4 : 1, 3, 5
5 : 0, 1, 2, 4

	Matrix	Adj. List
Edge (u, v)?	$O(1)$	$O(d)$
Neighbors of u	$O(\lvert V\rvert)$	$O(d)$
Space	$O(\lvert V\rvert^2)$	$O(\lvert E\rvert)$

Test your understanding..



Adjacency list of node 0?
(A) 0 : 1
(B) 0 : 1, 2
(C) 0 : 2
(C)
How many edges?
(A) 2
Total length of adacency lists?
(A) 2
(B) 3
(C) 4
(C) 2 entries for each edge!

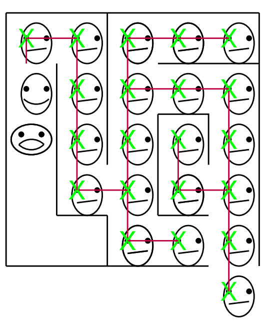
Exploring a maze.

Theseus: Wants to find the minatour in the maze.

Theseus has access to a Ball of Thread and a Chalk!

Explore a room: Mark room with chalk.
For each exit.
  Look through exit. If marked, next exit.
  Otherwise go in room unwind thread.
    Explore that room.
Wind thread to go back to "previous" room.

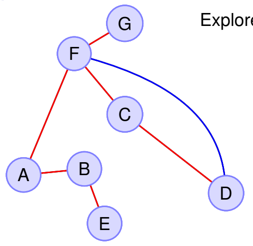
Where is the minatour?



Reachability problem in a Graph.

Problem: Find out which nodes are reachable from A.
Need digital analogues of the chalk and ball of thread.
We will use array (visited) for chalk and stack for thread.

Explore.



Explore(v):
1. Set visited[v] := true
2. for each edge (v,w) in E
3. if not visited[w]: Explore(w).

Chalk.
Stack is Thread.

Explore builds tree.
Tree and *back* edges.

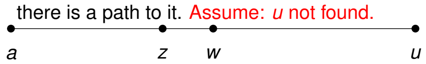
Correctness.

Explore(v):
1. Set visited[v] := **true**.
2. For each edge (v,w) in E
3. if not visited[w]: Explore(w)

Property:
All and only nodes reachable from A are reached by explore.

Only: when u visited.
stack contains nodes in a path from a to u.

All: if a node u is reachable.
there is a path to it. Assume: u not found.



z is explored. w is not!
Explore (z) would explore(w), or it was already explored!
Contradiction.

□

Running Time.

Explore(v):
1. Set visited[v] := **true**.
2. For each edge (v,w) in E
3. if not visited[w]: Explore(w).

How to analyse?

Let $n = |V|$, and $m = |E|$.

$T(n,m) \leq (d)T(n-1,m) + O(d)$ Exponential
?!?!?!

Don't use recurrence!

Running Time.

Explore(v):
1. Set visited[v] := **true**.
2. For each edge (v,w) in E
3. if not visited[w]: Explore(w).

How to analyse?

Let $n = |V|$, and $m = |E|$.

"Charge work to something."

For node x:
Explore once!
Process each incident edge.

Each edge processed twice.

$O(n)$ - call explore on n nodes.
$O(m)$ - process each edge twice.
Total: $O(n+m)$.

Depth first search.

Process whole graph.

DFS(G)
1: For each node u,
2: visited[u] = **false**.
3: For each node u,
4: if not visited[u] **explore(u)**

Running time: $O(|V|+|E|)$.

Intuitively: tree for each "connected component".
Several trees or Forest! Output connected components?

DFS and connected components.

Change explore a bit:

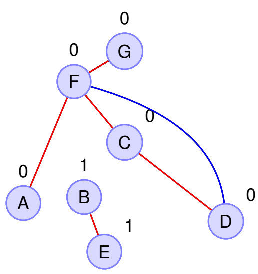
explore(v):
1. Set visited[v] := **true**.
2. previsit(v)
3. For each edge (v,w) in E
4. if not visited[w]: explore(w).
5. postvisit(v)

previsit(v):
1. Set cc[v] := ccnum.

DFS(G):
0. Set ccnum := 0.
1. for each v in V:
2. if not visited[v]:
3. explore(v)
4. ccnum = ccnum+1

Each node will be labelled with connected component number.
Runtime: $O(|V|+|E|)$.

Connected Components.



Introspection: pre/post.

previsit(v):
1. Set pre[v] := clock.
2. clock := clock+1

postvisit(v):
1. Set post[v] := clock.
2. clock := clock+1

DFS(G):
0. Set clock := 0.

$\vdots$

Clock: goes up to 2 times number of vertices.
First pre: 0

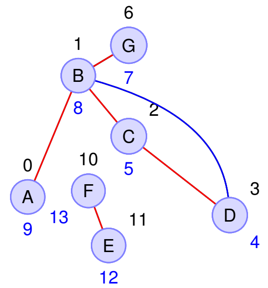
Property:
For any two nodes, u and v, $[pre(u), post(u)]$ and $[pre(v), post(v)]$ are either **disjoint** or **one is contained in other.**

Interval is "clock interval on stack."

Either both on stack at some point (contained) or not (disjoint.)

Let's just watch it work!

Example: Pre/Post numbering.

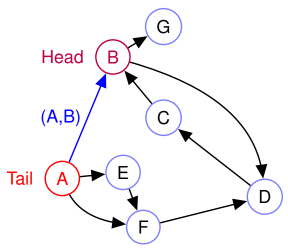


Edge (u, v) is tree edge iff $[pre[v], post[v]] \subset [pre[u], post[u]]$.
u on stack before v.
Edge (u, v) is back edge iff $[pre[u], post[u]] \subset [pre[v], post[v]]$.
v on stack before u on stack. Path from v to u! Cycle!
No edge between u and v if disjoint intervals.

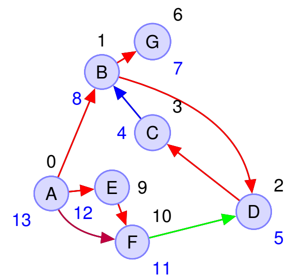
Directed graphs.

$G = (V, E)$
vertices V.
edges $E \subseteq V \times V$.

Edge: (u, v)
From u to v.
Tail – u
Head – v



Depth first search: directed.



Tree/forward edge (u, v): $int(v) \subset int(u)$. $inv(v) = [pre(v), post(v)]$
Forward (A, F): [10,11] in [0,13] or [0,[10,11],13]

Back edge (u, v): $int(u) \subset int(v)$.
(C, B): [3,4] in [1,8] or [1, [3, 4], 8]

Cross edge (u, v): $int(v) < int(u)$.
(F, D): [2,5] before [10,11]

Cycle in a directed graph?

Fast algorithm for finding out whether directed graph has cycle?

For each edge (u, v) remove, check if v is connected to u
$O(|E|(|E| + |V|))$.

Linear Time (i.e. $O(|V| + |E|)$)?

Testing for cycle.

Thm: A graph has a cycle if and only if there is a back edge in any DFS.

Proof:
We just saw: Back edge $\implies$ cycle!

In the other direction: Assume there is a cycle

$v_0 \to v_1 \to v_2 \cdots \to v_k \to v_0$

Assume that v_0 is the first node explored in the cycle
(without loss of generality since can renumber vertices.)

When **explore**(v_0) returns all nodes on cycle explored.

All int$[v_i]$ in int$[v_0]$!

$\implies (v_k, v_0)$ is a back edge.

Cycle $\implies$ back edge! $\square$

Fast checking algorithm.

Thm: A graph has a cycle if and only if there is back edge.

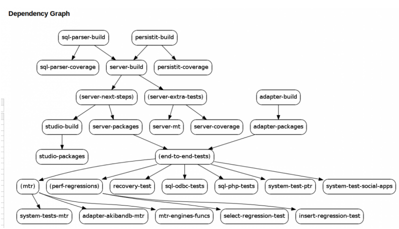
Algorithm ??

Run DFS.
$O(|V| + |E|)$ time.

For each edge (u, v): is int(u) in int(v)?
$O(|E|)$ time.

$O(|V| + |E|)$ time algorithm for checking if graph is acyclic!

Directed Acyclic Graph



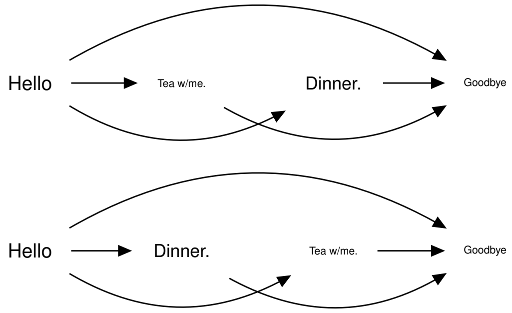
"Hello" before "Goodbye

No cycles! Can tell in linear time!

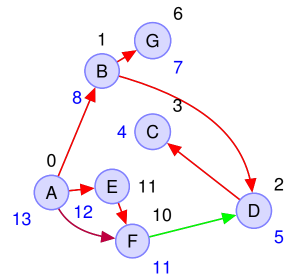
Really want to find ordering for build!

Linearize.

Topological Sort: For $G = (V, E)$, find ordering of all vertices where each edge goes from earlier vertex to later in acyclic graph.



Topological Sort Example.



A linear order:

A, E, F, B, G, D, C

In DFS: When is A popped off stack?

Last! When is E popped off? second to last. ...

Topological Sort: DFS

Last post order should..

(A) be first in linearization!

(B) be last in linearization!

(A). First!

Property: Every edge in a DAG (u, v) has $post(u) > post(v)$.

Proof: No back edges in DAG.
Tree and Forward edge (u, v):
int(u) contains int(v): $pre(u), pre[v], post[v], post[u]$

Cross edge (u, v): int$(u) >$ int$(v) \implies post[u] > post[v]$

□

Topological Sort: linearize.

Property: Every edge in a DAG (u, v) has $post(u) > post(v)$.

Top Sort: output in reverse post order number.

Runtime: $O(|V| + |E|)$.

..procedure **postvisit** outputs during DFS
def postvisit(u): result.append(u).
..reverse **result**.