\textbf{Prim}(G, w)

\forall u \in V \; \text{cost}(u) = \infty,

Pick any \( u_0 \in V \)

\text{cost}(u_0) = 0 \quad \text{prev}(u_0) = u_0

\forall v \in V \; \text{insert key}(v, \text{cost}(v))

while \text{queue non empty}

\( v = \text{Delete Min} \)

\forall \{v, u\} \in E

\text{if cost}(u) > w(v, u)

\text{cost}(u) = w(v, u)

\text{prev}(u) = v

\text{Decrease Key}(u)

\text{Output prev}[u] \; \forall u \in V

\textbf{Claim}: Interpreting prev(u) as the parent of u,

\text{Prim}(G, w) \; \text{outputs a MST rooted at } u_0

\textbf{Proof}: Let \( Q_t \) be the queue after \( t \) steps, with \( Q_0 = V \), and let \( S_t = V \setminus S_t \).

The algorithm starts with

\text{cost}(u_0) = 0, \; \text{cost}(u) = \infty \; \forall u \neq u_0

so after the first round \( S_1 = \{u_0\} \) and

\text{cost}(u) = w_{u_0 u}

\text{prev}(u) = u_0
For all $u \in Q_t$ s.th. $u \in Q_t$ is an edge from $S_t$ to $Q_t = V \setminus S_t$. So when we call $\text{DeleteMin}(Q_t)$ and set $v_t = \text{DeleteMin}(Q_t)$

$$w_{u,v_t} = \min_{u \in S_t} \min_{v \in V \setminus S_t} w_{uv} \quad u \in E$$

and the edge $u_v, v_t = \text{pre}(v_t) v_t$ is correctly added to the MST in accordance with the cut property.

The following claim ensures that holds for all steps

Claim: Just before the $\text{DeleteMin}$ is executed for the $t^{th}$ time the following two equalities hold for all $u \in Q_t$

$$\text{cost}(u) = \min_{v \in S_t} w_{vu} = w_{\text{pre}(u)} u$$

Proof by induction

Base case $t = 1$

Let $v_t = \text{DeleteMin} Q_t$

$$v_t$$

Let $\text{cost}(u) = \min_{v \in S_t} w_{vu} = w_{\text{pre}(u)} u$ at end of last round

Updating $S_t$ gives $S_{t+1} = S_t \cup v_t$.

If $w_{u,v_t} < \text{cost}(u)$ we correctly update both
cost(u) to \( v_{\text{rel}} \) and \( \text{prev}(u) = v_b \), see figure.

Otherwise, no update is needed.