Linear Programming

Q: What is a linear program

A: An optimization problem with

linear objective, and
linear constraints

e.g. max $x+2y$
$x, y \geq 0$ $x+y \leq 1$

More general: n variable, m constraints

Example: Christian's Bakery

Menu: Donut \$5
Cake \$25

Recipies:

Ingredients	Donut	Cake
Flour (200 total)	2	5
Suggar (300 total)	2	9
Eggs (500 total)	7	12

Q: How many donuts & cakes should I make to maximize profit

Linear Program (LP)

Decision Variables

x # of donuts y # of cakes

Constraints

$x, y \geq 0$

$2x + 5y \leq 200$

$2x + 9y \leq 300$

$7x + 12y \leq 500$

Objective

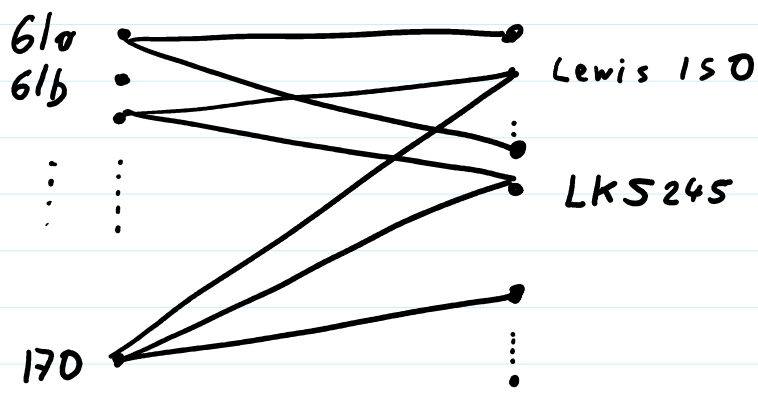
Maximize $5x + 25y$

Notes:

- Why **Linear** Program: Constraints & Objective are Linear
- Why Linear **Program**: Danzig, 1947, US Airforce: used to optimize resource allocation. At the time, **schedules** were called **Programs**
- LP vs Integer LP

Ex 2: Classroom Allocation

Courses Classrooms



Class c can be assigned to room r

$\iff (c, r) \in E$

Goal: maximize # of courses assigned to a room

LP

Variables

$x_{c,r} = 1 \iff$ class c is assigned to room r

Constraints

$\forall c, r \in E \quad x_{c,r} \geq 0$

$\forall r \quad \sum_{c:(c,r)\in E} x_{c,r} \leq 1$

$\forall c \quad \sum_{r:(c,r)\in E} x_{c,r} \leq 1$

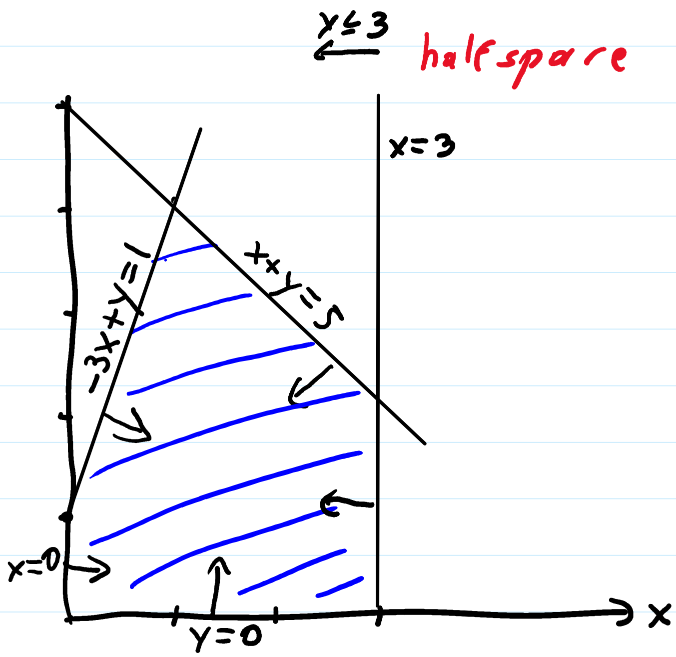
Objective

Maximize $\sum_{c,r \in E} x_{c,r}$

How to solve an LP

1) Feasible Region

$$\max \ x + 2y$$
subject to: $x, y \geq 0$
$x \leq 3$
$x + y \leq 5$
$-3x + y \leq 1$

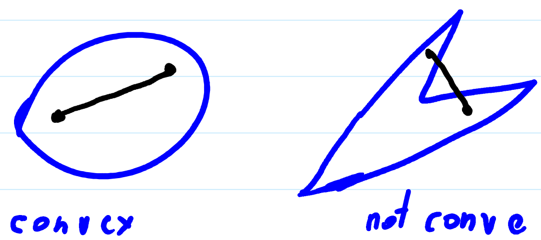


Def: Feasible region = set of points satisfying all constraints

Fact: Feasible region is convex

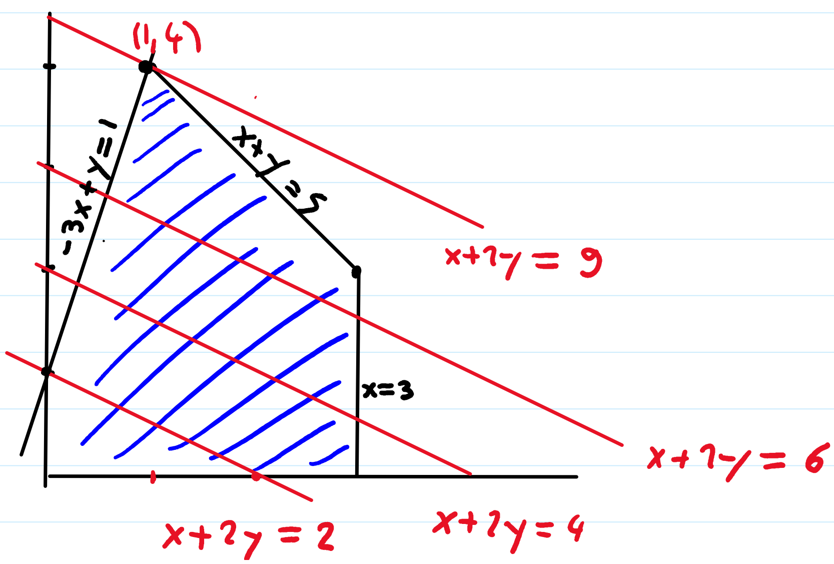
Def.: S is convex

$\Leftrightarrow \forall x, y \in S$ line $x \to y \subseteq S$



2) How to find the maximum

A: Look at Levelsets $x + 2y = c$



$$\max \ x + 2y$$
subject to: $x, y \geq 0$
$x \leq 3$
$x + y \leq 5$
$-3x + y \leq 1$

Claim: If feas. region is bounded & non-empty

$\Rightarrow$ $\exists$ optimum at vertex of polytope

3) Proof that $x+2y=9$ is optimal

- max ≥ 9 since there is a feasible point with that value
- max ≤ 9: Any feasible point satisfies

$$\left.\begin{array}{rl} x+y \leq 5 & \cdot \frac{7}{4} \\ -3x+y \leq 1 & \cdot \frac{1}{4} \end{array}\right\}$$

$$\overline{\quad x+2y \leq 5 \quad}$$

Where do the magic numbers 7/4, 1/4 come from?

A: LP-Duality

General Linear Programm

n variables $x_1, \cdots, x_n$

m constraints: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i=1,\cdots,m)$

Objective: $\max \sum_{i=1}^{n} c_i x_i$

Variants

$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$

$\sum_{j=1}^{n} a_{ij} x_j = b_i$

$\min \sum_{i=1}^{n} c_i x_i$

The variants are equivalent

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \iff \sum_{j=1}^{n} (-a_{ij}) x_j \leq -b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \iff \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ and } \sum_{j=1}^{n} -a_{ij} x_j \leq -b_i$$

$$\min \sum_{i=1}^{n} c_i x_i = \max \sum_{i=1}^{n} (-c_i) x_i$$

General LP

n variables $x_1, \cdots, x_n$

m constraints: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$

Objective: $\max \sum_{i=1}^{n} c_i x_i$

Matrix Form

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

maximize $c^T x$

subject to $Ax \leq b$

Canonical Form

$$\max c^T x \quad \text{s.t.} \quad x \geq 0 \quad Ax \leq b$$

Claim: This is another equivalent form

1) Transformation of the canonical form to the general form: If x solves $\max c^T x$ s.th. $x \geq 0$ and $Ax \leq b$, set

- $\tilde{A} = \begin{bmatrix} A \\ -I \end{bmatrix}$ where I is the $n \times n$ identity matrix, $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$
- $\tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} \in \mathbb{R}^{m+n}$

Then $\tilde{A}x = \begin{bmatrix} Ax \\ -x \end{bmatrix}$ and $\tilde{A}x \leq \tilde{b}$ is equivalent to $x \geq 0$, $Ax \leq b$ ✓

2) Transforming our general form to the canonical form

If x solves $\max c^T x$ s.t. $Ax \le b$, write $x \in \mathbb{R}^n$ as

$x = x_+ - x_-$, $x_+, x_- \in \mathbb{R}^n_+$, and set

$$\tilde{x} = \begin{pmatrix} x_+ \\ x_- \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} c \\ -c \end{pmatrix} \in \mathbb{R}^{2n}, \quad \tilde{A} = [A, -A]$$

Then

$$\tilde{c}^T \tilde{x} = c^T(x_+ - x_-) = c^T x, \quad \tilde{A}\tilde{x} = Ax_+ - Ax_- = Ax$$

Thus

$\max c^T x$ s.t. $Ax \le b$ is equivalent to the $2n$ dimensional linear program

$\max \tilde{c}^T \tilde{x}$ s.th $\tilde{x} \ge 0$ and $\tilde{A}\tilde{x} \le b$

Algorithms for solving an LP

Recall: For $n=2$, there exist optimum that is a vertex of the Polytop

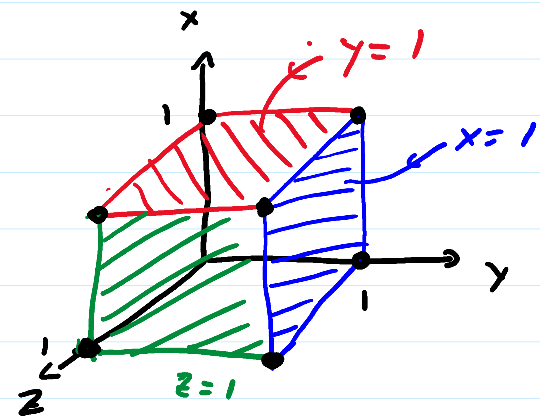
Q: What is a vertex for general n?

Ex: max $x+y+z$

s.t. $0 \le x \le 1$

$0 \le y \le 1$

$0 \le z \le 1$



vertex = intersection of 3 planes

—— " —— 3 constraints

n variables, m constraints:

Def: A vertex x is an intersection of n constraints s.th. x lies in the polytope (= feas. region)

Fact: There exist optimum which is a vertex

Naive Alg: Look at all vertices

Claim: # of vertices $\le \binom{m}{n}$ need to choose n constraints out of m

LP-Algorithm #1

Input $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

Set Opt $= -\infty$

For each $C \subseteq \{1, \dots m\}$, $|C| = n$

- Solve $\sum_{j=1}^{n} A_{ij} x_j^* = b_i$ for all $i \in C$ ← Gaussian Elimination
- If $Ax^* \le b$

  If $c^T x^* > Opt$ set $Opt = c^T x^*$

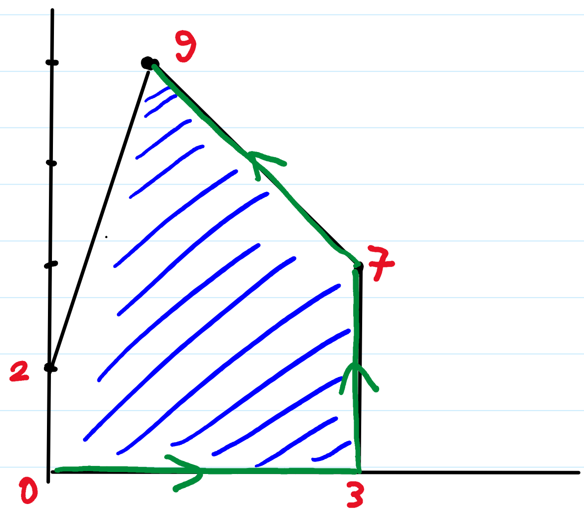
Output Opt

Run Time $\cong \binom{m}{n} n^3$ $m=40, n=20 \rightarrow 10^{15}$ (20min on Intel core I7)

Gaussian Elimination

Simplex Algorithm

- Start at vertex x^*
- Find neighbor y^* with maximal value
- If value$(y^*) >$ value(x^*) move to y^*. Repeat



Q: What is a neighbor?

Vertex = Intersection of n constraints

Neighbors differ in 1 constraints

Q: How many neighbors are there?

Ex: $n=3$ Constraints $\{A, B, C, D, E\}$ $(m=5)$

Vertex ACD

Neighb. ABC, ACE, ABD, ADE, BCD, CDE

Q: Upper Bd on # of neighbors? $n \cdot m$

Sad Fact: Still exponential in worst case

Good News: Fast in practice

Theory: Smooth Analysis: mathematical proof if we "perturb" a worst case

Standard Form of an LP

When analysing LP's it is sometimes useful to consider yet another equivalent form, called the standard form

$$\max c^T x \text{ subject to } x \geq 0,\ Ax = b \qquad (S)$$

Claim: This is equivalent to the other forms, e.g., the canonical form

$$\max c^T x \quad \text{s.t.} \quad x \geq 0 \quad Ax \leq b \qquad (C)$$

Transforming S to C

Define $A' = \begin{pmatrix} A \\ -A \end{pmatrix}$ $b' = \begin{pmatrix} b \\ -b \end{pmatrix}$. Then

$$Ax = b \Leftrightarrow A'x = \begin{pmatrix} Ax \\ -Ax \end{pmatrix} \leq \begin{pmatrix} b \\ -b \end{pmatrix}$$

Transforming C to S

Consider a solution x to the canonical LP

$$\max \ c^T x \quad \text{s.t.} \quad x \geq 0 \quad Ax \leq b$$

Define the "slack variables"

$$s_i = b_i - (Ax)_i$$

Then $s_i, x_j \geq 0$ and $(s + Ax) = b$

Set

$$\tilde{x} = \begin{pmatrix} x \\ s \end{pmatrix} \in \mathbb{R}^{n+m} \qquad \tilde{c} = \begin{pmatrix} c \\ 0 \end{pmatrix} \qquad \tilde{A} = [A, I]$$

Then the canonical LP is equivalent to

$$\max \ \tilde{c}^T \tilde{x} \quad \text{s.t.} \quad \tilde{A}\tilde{x} = b \qquad \blacksquare$$

Other Algorithms:

Thm: LPs can be solved in polynomial time

[Khachiyan, 1979] Ellipsoid Alg.

[Karmarkar, 1984] Interior Point Method


The New York Times

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A Soviet Discovery Rocks World of Mathematics

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

Mathematicians describe the discovery by L.G. Khachian as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of hit-or-miss basis.

Apart from its profound theoretical interest, the discovery may be applicable in weather prediction, complicated industrial processes, petroleum refining, the scheduling of workers at large factories, secret codes and many other things.

"I have been deluged with calls from virtually every department of government for an interpretation of the significance of this," a leading expert on computer methods, Dr. George B. Dantzig of Stanford University, said in an interview.

The solution of mathematical problems by computer must be broken down into a series of steps. One class of problem sometimes involves so many steps that it could take billions of years to compute.

The Russian discovery offers a way by which the number of steps in a solution can be dramatically reduced. It also offers the mathematician a way of learning quickly whether a problem has a solution or not, without having to complete the entire immense computation that may be required.

According to the American journal Sci-

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The New York Times

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Breakthrough in Problem Solving

Homeless Spend Nights in City Welfare Office

High Cuban Defector Speaks Out, Denouncing Castro as 'Impulsive'

Breakthrough in Problem Solving

By JAMES GLEICK

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

These problems are fiendishly complicated systems, often with thousands of variables. They arise in a variety of commercial and government applications, ranging from allocating time on a communications satellite to routing millions of telephone calls over long distances. Linear programming is particularly useful whenever a limited, expensive resource must be spread most efficiently among competing users. And investment companies use the approach in creating portfolios with the best mix of stocks and bonds.

Faster Solutions Seen

The Bell Labs mathematician, Dr. Narendra Karmarkar, has devised a radically new procedure that may speed the routine handling of such problems by businesses and Government agencies and also make it possible to tackle problems that are now far out of reach.

"This is a path-breaking result," said Dr. Ronald L. Graham, director of mathematical sciences for Bell Labs in Murray Hill, N.J. "Science has its moments of great progress, and this may well be one of them."

Because problems in linear programming can have billions or more possible answers, even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one — typically the one that minimizes cost or maximizes efficiency.

A procedure devised in 1947, the simplex method, is now used for such prob-

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Homeless Spen

By SARA RIMER

For the last 10 weeks, homeless families, mostly mothers and young children, have been spending weekend nights on plastic chairs, on countertops or on the floor in New York City's emergency welfare office because the city's welfare agency has run out of beds.

Other families have been waiting almost through the night while city welfare workers try to find temporary space for them in any of the 51 hotels scattered throughout Manhattan, the Bronx, Brooklyn and Queens that accept homeless families.

In some cases, the families leave the Manhattan office at 4 or even 5 A.M. for an hour's trip on the subway to hotels in the other boroughs that will require them to check out as early as 11 A.M. that same morning.