Minimum spanning tree.
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CS 170: Algorithms
Minimum spanning tree.
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Lecture in a minute.

Tree Definitions:
- $n - 1$ edges and connected.
- $n - 1$ edges and no cycles.
- All pairs of vertices connected by unique path.
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Minimum Spanning Tree: $G = (V, E)$, weights $w : E$

Kruskal: Sort edges.
- Add edges in this order if no cycle.

Cut property:
- Exists MST with minimum weight edge across cut.

Union-Find Data Structure.

Pointer implementation:
- $\pi(u)$.
- $\text{makeset}(s)$ – $\pi(u) = u$.
- $\text{find}(x)$ – returns root of pointer structure.
- $\text{union}(x, y)$ – $\pi(\text{find}(x)) = \pi(\text{find}(y))$.

Union by rank:
- $O(\log n)$ depth for pointer structure.
- $\text{union}(x, y)$ – point to larger rank root.
- Increase rank if tied.
- $\geq 2^k$ nodes in rank $k$ root tree.

$O(\log n)$ depth structure.
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Trees.

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Start with empty graph with $n$ components.
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Adding any edge between components reduces the number of components by one. After \( n - 1 \) additions, one component! (If more additions, inside component will create a cycle.)
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```
A ---- E
   \   \   \   \\
B     C     D
      \   |
       \ / \\
      A
```

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Remove edge on cycle, still connected.
Def: A tree is a connected graph with no cycles.
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If not, there is \( n - 1 \) edge connected graph with a cycle.

Remove edge on cycle, still connected. And \( n-2 \) edges.
Def: A tree is a connected graph with no cycles.

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Remove edge on cycle, still connected. And $n - 2$ edges. Must have at least $n - 1$ edges to be connected.
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**Def:** A tree is a connected graph with no cycles.
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If two paths:

\[ u \quad v \]

Diverge

Come back together.

\[ \Rightarrow \] cycle!

Not Tree!

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Minimum Spanning Tree.

Given a graph, $G = (V, E)$, edge weights $w_e$, find the cheapest possible connected subgraph.
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Will it be a tree?
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Yes? No?

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Yes? No?

Yes. If edge weights positive.
Minimum Spanning Tree.

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Will it be a tree?

Yes? No?

Yes. If edge weights positive.

If negative edges, then restrict to tree.
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If negative edges, then restrict to tree.

Given a graph, $G = (V, E)$, edge weights $w_e$, find the lowest weight spanning tree.
To MST or not!

- MST - cheapest spanning tree of graph.
- Shortest path tree - contains shortest paths from $s$ to other nodes.
- MST - do not care about shortest paths! just lowest weight tree.
To MST or not!

Shortest Path Tree from $s$ to $v$.

MST - cheapest spanning tree of graph.

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MST?    MST?

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Shortest Path Tree from s!

MST? Yes!

Shortest path from s to v in tree?
To MST or not!

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Example and Algorithm

MST: total cost is $2 + 4 + 3 + 1 + 5 = 15$. 
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Diagram showing a graph with nodes A, B, C, D, E, and F, with edges and weights as follows:
- A to B: 5
- B to C: 3
- B to E: 3
- C to D: 4
- C to F: 6
- D to F: 5
- E to F: 1
Example and Algorithm

MST: total cost is $2 + 4 + 3 + 1 + 5 = 15$. 
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Smallest edge across any cut is in some MST.
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$S$  $V - S$

![Diagram](image)
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Tree Connected $\Rightarrow$
Cut property.

Smallest edge across any cut is in some MST.

Tree Connected $\implies$ there exists $e'$ across cut! Replace $e'$ with $e$. 
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Tree Connected $\implies$ there exists $e'$ across cut! Replace $e'$ with $e$. Every pair remains connected.
Cut property.

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Tree Connected $\iff$ there exists $e'$ across cut! Replace $e'$ with $e$.
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If used $e'$ can use path through $e$. 
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Tree Connected $\implies$ there exists $e'$ across cut! Replace $e'$ with $e$.
Every pair remains connected.
If used $e'$ can use path through $e$.
and $n-1$ edges.
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\[ S \quad e \quad V - S \]

Tree Connected \( \implies \) there exists \( e' \) across cut! Replace \( e' \) with \( e \).

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If used \( e' \) can use path through \( e \).

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So still a tree
Cut property.

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$S$ $V - S$

Tree Connected $\implies$ there exists $e'$ across cut! Replace $e'$ with $e$.

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So still a tree and is no more costly ($w(e) \leq w(e')$.)
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Tree Connected \( \implies \) there exists \( e' \) across cut! Replace \( e' \) with \( e \).
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So still a tree and is no more costly \((w(e) \leq w(e')).\)
Sort edges.

\[ F = . \text{ For each edge: } e \]

\[
\text{If no cycle, } F = F + e. \]
Kruskal

Sort edges.

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How to check for cycle for edge \((u, v)\) in \(F\)?
Kruskal

Sort edges.
$F = \emptyset$. For each edge: $e$
   If no cycle, $F = F + e$.

How to check for cycle for edge $(u, v)$ in $F$?
Check for path between $u$ and $v$ in $F$. 
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Check for path between \( u \) and \( v \) in \( F \).

Total Running time?
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\[ \begin{align*}
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\end{align*} \]

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Total Running time?

\( O(n) \) time
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**How to check for cycle for edge \((u, v)\) in \( F \)?**
Check for path between \( u \) and \( v \) in \( F \).

**Total Running time?**
\( O(n) \) time \( \rightarrow \) \( O(nm) \) for Kruskals.
Kruskal

Sort edges.

$F = \emptyset$. For each edge: $e = (u, v)$

If no cycle in $F$, add edge.
Kruskal

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Main issue: Check for cycle.
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Maintain connected components.
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Maintain connected components.

At beginning each node by itself.
Kruskal

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Maintain connected components.

At beginning each node by itself.

Adding edge, joins component.
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Edge \((u, v)\) in cycle?
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Sort edges.
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Disjoint Sets Data Structure.
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Adding edge, joins component.
Edge \((u, v)\) in cycle? \( u \) and \( v \) in same component.

Disjoint Sets Data Structure.

\texttt{makeset}(x) - makes singleton set \( \{x\} \).
Kruskal

Sort edges.
\( F = \). For each edge: \( e = (u, v) \)
   - If no cycle in \( F \), add edge.

Main issue: Check for cycle.

Maintain connected components.

At beginning each node by itself.
Adding edge, joins component.
Edge \((u, v)\) in cycle? \( u \) and \( v \) in same component.

Disjoint Sets Data Structure.

\texttt{makeset}(x) - makes singleton set \( \{x\} \).
\texttt{find}(x) - finds set containing \( x \).
Kruskal

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\[ F = \ldots \]

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Disjoint Sets Data Structure.

makeSet($x$) - makes singleton set $\{x\}$.
find($x$) - finds set containing $x$.
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“If no cycle” $\equiv$ “find($u$) $\neq$ find($v$)”

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Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. 

How long does find take?

(A) $O(n)$

(B) $O(1)$

(C) Depends.

Want depth to be small!
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$.

makeset($x$)

How long does find take? (A) $O(n)$ (B) $O(1)$ (C) Depends. Want depth to be small!
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$.

**makeset(x)** $\pi(x) = x$.

**union(x,y)**

$\pi(\text{find}(x)) = \text{find}(y)$

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$\pi(\text{find}(x)) = \text{find}(y)$

**find(x)**
if $\pi(x) == x$
return $x$
else
find($\pi(x)$)

How long does find take?

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- if \( \pi(x) == x \)
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\textbf{makeset}(x) $\pi(x) = x$.

\textbf{union}(x, y) $\pi(\text{find}(x)) = \text{find}(y)$

\textbf{find}(x)
  \begin{align*}
    &\text{if } \pi(x) == x \\
    &\quad \text{return } x \\
    &\text{else} \\
    &\quad \text{find}(\pi(x))
  \end{align*}

How long does \textbf{find} take?

(A) $O(n)$
(B) $O(1)$
(C) Depends.

Want depth to be small!
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. 
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`makeset(x)`
Disjoint Set Data Structure

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Disjoint Set Data Structure

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**makeset(x)** \( \pi(x) = x \).

**find(x)**
- if \( \pi(x) == x \)
  - return \( x \)
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Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$.

**makeset**(x) $\pi(x) = x$.

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  if $\pi(x) == x$
   
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  else
   
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Make a bit less deep: union-by-rank.
Maintain pointers: $\pi(x)$ for each $x$.

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Make a bit less deep: union-by-rank.

union(x,y)
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$.

**makeset(x) $\pi(x) = x$.**

**find(x)**

- if $\pi(x) == x$
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Make a bit less deep: union-by-rank.

**union(x,y)**

Use roots of $x$ and $y$. 
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$.

**makeset(x)** $\pi(x) = x$.

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Make a bit less deep: union-by-rank.

union(x,y)

Use roots of x and y.

Which points to which?
Maintain pointers: $\pi(x)$ for each $x$.

```makefile
makeset(x) $\pi(x) = x$.
```

```makefile
find(x)
   if $\pi(x) == x$
      return x
   else
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```

Make a bit less deep: union-by-rank.

```makefile
union(x,y)
Use roots of $x$ and $y$.
Which points to which?
“smaller” to “larger”```
Disjoint Set Data Structure

Maintain pointers: \( \pi(x) \) for each \( x \).

\textbf{makeset}(x) \ \ \ \pi(x) = x.

\textbf{find}(x)
  \quad \text{if} \ \ \pi(x) == x
  \quad \text{return} \ x
  \quad \text{else}
  \quad \quad \text{find}(\pi(x))

Make a bit less deep: union-by-rank.

union(x,y)
Use roots of \( x \) and \( y \).
Which points to which?
“smaller” to “larger” ..sort of.
Union by rank.

Initially: \( \text{rank}(x) = 0. \)
Union by rank.

Initially: $\text{rank}(x) = 0$.

**union**(x,y)

$r_x = \text{find}(x)$

$r_y = \text{find}(y)$
Union by rank.

Initially: \( \text{rank}(x) = 0 \).

\textbf{union}(x,y)
\begin{itemize}
    \item \( r_x = \text{find}(x) \)
    \item \( r_y = \text{find}(y) \)
    \item \textbf{if} \( \text{rank}(r_x) < \text{rank}(r_y) \): \\
        \( \pi(r_x) = r_y \)
    \item \textbf{if} \( \text{rank}(r_x) == \text{rank}(r_y) \):
        \( \text{rank}(r_x) += 1 \)
\end{itemize}
Initially: \( \text{rank}(x) = 0 \).

\[ \text{union}(x,y) \]
\[ r_x = \text{find}(x) \]
\[ r_y = \text{find}(y) \]
\[ \text{if } \text{rank}(r_x) < \text{rank}(r_y): \]
\[ \pi(r_x) = r_y \]
Union by rank.

Initially: \( \text{rank}(x) = 0 \).

**union(x, y)**

- \( r_x = \text{find}(x) \)
- \( r_y = \text{find}(y) \)

  - **if** \( \text{rank}(r_x) < \text{rank}(r_y) \):
    - \( \pi(r_x) = r_y \)
  
  - **else**:
    - \( \pi(r_y) = r_x \)
Union by rank.

Initially: rank(x) = 0.

union(x,y)
  \[ r_x = \text{find}(x) \]
  \[ r_y = \text{find}(y) \]
  \textbf{if} rank(r_x) < rank(r_y): \[
  \pi(r_x) = r_y
  \]
  \textbf{else:} \[
  \pi(r_y) = r_x
  \]
  \textbf{if} rank(r_x) == rank(r_y): \[
  \text{rank}(r_x) += 1
  \]
Why rank?

**Lemma:** Pop’s got a higher rank:
Why rank?

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\[
\text{rank}(x) < \text{rank}(\pi(x))
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Duh!
Code enforces it.
union(x,y):
  if rank(r_{x}) < rank(r_{y}):
    \pi(r_{x}) = r_{y}
  else:
    \pi(r_{y}) = r_{x}
if rank(r_{x}) == rank(r_{y}):
  rank(r_{x}) += 1
Why rank?

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**Lemma:** Pop’s got a higher rank:

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Code enforces it.

```python
union(x, y):
    :
    if \text{rank}(r_x) < \text{rank}(r_y):
        \pi(r_x) = r_y
    else:
        \pi(r_y) = r_x
    if \text{rank}(r_x) == \text{rank}(r_y):
        \text{rank}(r_x) += 1
```
Why rank?

**Lemma:** Pop’s got a higher rank:

\[ \text{rank}(x) < \text{rank}(\pi(x)) \]

if \( x \neq \pi(x) \).

Duh!

Code enforces it.

\begin{verbatim}
union(x,y):
  :
  if rank(r_x) < rank(r_y):
    \pi(r_x) = r_y
  else:
    \pi(r_y) = r_x
  if rank(r_x) == rank(r_y):
    rank(r_x) += 1
\end{verbatim}

Initially?
Big rank is a big dog!
Big rank is a big dog!

union(x,y):

if rank(r_x) < rank(r_y):
    \[ \pi(r_x) = r_y \]
else:
    \[ \pi(r_y) = r_x \]
    if rank(r_x) == rank(r_y):
        rank(r_x) += 1

Lemma: Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.
Big rank is a big dog!

union(x,y):
  :
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  if rank(r_x) < rank(r_y):
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**Lemma:** Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

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Big rank is a big dog!

union(x,y):
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  if rank(r_x) == rank(r_y):
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**Lemma:** Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

**Induction:**

**Base Case**
union(x,y):
  
  if rank(r_x) < rank(r_y):
    π(r_x) = r_y
  else:
    π(r_y) = r_x
    if rank(r_x) == rank(r_y):
      rank(r_x) += 1

**Lemma:** Any rank $k$ root node has $\geq 2^k$ nodes in its tree.

**Induction:**

**Base Case ?**
union(x, y):

if \( \text{rank}(r_x) < \text{rank}(r_y) \):
  \( \pi(r_x) = r_y \)
else:
  \( \pi(r_y) = r_x \)
  if \( \text{rank}(r_x) == \text{rank}(r_y) \):
    \( \text{rank}(r_x) += 1 \)

**Lemma:** Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

**Induction:**

**Base Case?**

(A) \( 2^0 \geq 1 \)

(B) \( 2^1 \geq 1 \)
Big rank is a big dog!

union(x,y):
  
  if rank(r_x) < rank(r_y):
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      rank(r_x) += 1

Lemma: Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

Induction:
Base Case?

(A) \( 2^0 \geq 1 \)

(B) \( 2^1 \geq 1 \)

A.
Big rank is a big dog!

union(x,y):
    :
    :
    if rank(r_x) < rank(r_y):
        π(r_x) = r_y
    else:
        π(r_y) = r_x
        if rank(r_x) == rank(r_y):
            rank(r_x) ++ = 1

Lemma: Any rank $k$ root node has $\geq 2^k$ nodes in its tree.
Induction:
Base Case?

(A) $2^0 \geq 1$
(B) $2^1 \geq 1$

A. Initially $rank(x) = 0$, 1 node in tree.
Big rank is a big dog!

union(x,y):

if rank(r_x) < rank(r_y):
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(A) \( 2^0 \geq 1 \)

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A. Initially \( rank(x) = 0 \), 1 node in tree.

Induction step:
Big rank is a big dog!

union(x,y):
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    if rank(r_x) < rank(r_y):
        \( \pi(r_x) = r_y \)
    else:
        \( \pi(r_y) = r_x \)
        if rank(r_x) == rank(r_y):
            rank(r_x) + = 1

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**Base Case?**

\( \text{(A) } 2^0 \geq 1 \)

\( \text{(B) } 2^1 \geq 1 \)

A. Initially \( rank(x) = 0 \), 1 node in tree.

**Induction step:**

When \( rank(x) \) goes up to \( k \).
Big rank is a big dog!

union(x,y):
  ...
  if rank(r_x) < rank(r_y):
    \( \pi(r_x) = r_y \)
  else:
    \( \pi(r_y) = r_x \)
    if rank(r_x) == rank(r_y):
      rank(r_x) += 1

**Lemma:** Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

**Induction:**

**Base Case**

(A) \( 2^0 \geq 1 \)

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A. Initially \( \text{rank}(x) = 0 \), 1 node in tree.

**Induction step:**
When \( \text{rank}(x) \) goes up to \( k \).
  \( \text{rank}(x) \) was \( k - 1 \)
Big rank is a big dog!

union(x,y):

if rank(r_x) < rank(r_y):
    \( \pi(r_x) = r_y \)
else:
    \( \pi(r_y) = r_x \)
    if rank(r_x) == rank(r_y):
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Lemma: Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

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Base Case?

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A. Initially \( rank(x) = 0 \), 1 node in tree.

Induction step:
When \( rank(x) \) goes up to \( k \).
    \( rank(x) \) was \( k - 1 \) so has \( \geq 2^{k-1} \) nodes.
Big rank is a big dog!

union(x,y):
  
  if rank(r_x) < rank(r_y):
    \( \pi(r_x) = r_y \)
  else:
    \( \pi(r_y) = r_x \)
    if rank(r_x) == rank(r_y):
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Lemma: Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

Induction:
Base Case:

(A) \( 2^0 \geq 1 \)

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A. Initially \( rank(x) = 0 \), 1 node in tree.

Induction step:
When \( rank(x) \) goes up to \( k \).
  \( rank(x) \) was \( k - 1 \) so has \( \geq 2^{k-1} \) nodes. by ind. hyp.
Big rank is a big dog!

union(x,y):

... if rank(r_x) < rank(r_y):
   π(r_x) = r_y
else:
   π(r_y) = r_x
   if rank(r_x) == rank(r_y):
      rank(r_x) += 1

Lemma: Any rank $k$ root node has $\geq 2^k$ nodes in its tree.

Induction:
Base Case?

(A) $2^0 \geq 1$

(B) $2^1 \geq 1$

A. Initially $rank(x) = 0$, 1 node in tree.

Induction step:
When $rank(x)$ goes up to $k$.
   rank(x) was $k - 1$ so has $\geq 2^{k-1}$ nodes. by ind. hyp.
   gains nodes from rank $k - 1$ node
Big rank is a big dog!

union(x,y):

  ...
  if rank(r_x) < rank(r_y):
    \( \pi(r_x) = r_y \)
  else:
    \( \pi(r_y) = r_x \)
    if rank(r_x) == rank(r_y):
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**Lemma**: Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

**Induction**:

**Base Case**

(A) \( 2^0 \geq 1 \)

(B) \( 2^1 \geq 1 \)

A. Initially \( \text{rank}(x) = 0 \), 1 node in tree.

Induction step:

When \( \text{rank}(x) \) goes up to \( k \).

\( \text{rank}(x) \) was \( k - 1 \) so has \( \geq 2^{k-1} \) nodes. by ind. hyp.

\( \text{gains nodes from rank } k - 1 \) node with \( \geq 2^{k-1} \) nodes
Big rank is a big dog!

union(x, y):
  :
  :
  if rank(r_x) < rank(r_y):
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(A) \( 2^0 \geq 1 \)

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When \( rank(x) \) goes up to \( k \).
  \( rank(x) \) was \( k - 1 \) so has \( \geq 2^{k-1} \) nodes. by ind. hyp.
  gains nodes from rank \( k - 1 \) node with \( \geq 2^{k-1} \) nodes
  \( \implies \geq 2^{k-1} + 2^{k-1} = 2^k \) nodes.
Big rank is a big dog!

union(x,y):

\begin{verbatim}
if rank(r_x) < rank(r_y):
    \pi(r_x) = r_y
else:
    \pi(r_y) = r_x
    if rank(r_x) == rank(r_y):
        rank(r_x) += 1
\end{verbatim}

**Lemma:** Any rank \( k \) root node has \( \geq 2^k \) nodes in its tree.

**Induction:**

**Base Case:**

(A) \( 2^0 \geq 1 \)

(B) \( 2^1 \geq 1 \)

A. Initially \( \text{rank}(x) = 0 \), 1 node in tree.

**Induction step:**

When \( \text{rank}(x) \) goes up to \( k \).

- \( \text{rank}(x) \) was \( k - 1 \) so has \( \geq 2^{k-1} \) nodes. by ind. hyp.
- gains nodes from rank \( k - 1 \) node with \( \geq 2^{k-1} \) nodes

\( \implies \geq 2^{k-1} + 2^{k-1} = 2^k \) nodes. \( \square \)
Check your understanding?

Exactly $2^k$ nodes in tree of rank $k$?
Check your understanding?

Exactly $2^k$ nodes in tree of rank $k$? Yes?

Yes? 

No.

if rank($r_x$) < rank($r_y$):
π($r_x$) = $r_y$ ...

Gains nodes without gaining rank!
Check your understanding?

Exactly $2^k$ nodes in tree of rank $k$? Yes? No?
Check your understanding?

Exactly $2^k$ nodes in tree of rank $k$? Yes? No?
No.
Check your understanding?

Exactly $2^k$ nodes in tree of rank $k$? Yes? No?

No.

::

    if rank($r_x$) < rank($r_y$):
        $\pi(r_x) = r_y$

::
Check your understanding?

Exactly $2^k$ nodes in tree of rank $k$? Yes? No?
No.

::

    if rank($r_x$) < rank($r_y$):
        $\pi(r_x) = r_y$

::

Gains nodes without gaining rank!
Back to complexity.

Find(x) is

(A) $O(\log n)$ time.
(B) $O(1)$ time
(C) $O(n)$ time.
Back to complexity.

Find(x) is

(A) $O(\log n)$ time.

(B) $O(1)$ time

(C) $O(n)$ time.

A.
Back to complexity.

Find(x) is

(A) $O(\log n)$ time.

(B) $O(1)$ time

(C) $O(n)$ time.

A. (and (C)).
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Can we do better? Yes. We will see better.
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Kruskal Implementation.

$|V|$ unions. $|E|$ finds.
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$O(|E| \log n)$ time!
Kruskal Implementation.

$|V|$ unions. $|E|$ finds.

$O(|E| \log n)$ time!

Versus $O(|E||V|)$. 
Lecture in a minute.

Tree Definitions:

\( n - 1 \) edges and connected.
\( n - 1 \) edges and no cycles.
All pairs of vertices connected by unique path.
Lecture in a minute.

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Minimum Spanning Tree: \(G = (V, E)\), weights \(w : E\)

Kruskal: Sort edges.
  - Add edges in this order if no cycle.

Cut property:
  - Exists MST with minimum weight edge across cut.

Union-Find Data Structure.
- Pointer implementation: \(\pi(u)\).
- \(\text{makeset}(s)\) – \(\pi(u) = u\).
- \(\text{find}(x)\) – returns root of pointer structure.
- \(\text{union}(x,y)\) – \(\pi(\text{find}(x)) = \pi(\text{find}(y))\).

Union by rank:
- \(O(\log n)\) depth for pointer structure.
- \(\text{union}(x,y)\) - point to larger rank root.
  - Increase rank if tied.
  - \(\geq 2^k\) nodes in rank \(k\) root tree.
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Union by rank:
- $O(\log n)$ depth for pointer structure.
- $\pi(x) = \pi(y)$ for $\pi(x) \geq 2^{\pi(y)}$ nodes.
Lecture in a minute.

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See you on Wednesday!