CS 170: Algorithms
Prims and Huffman Coding.
CS 170: Algorithms

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CS 170: Algorithms

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CS 170: Algorithms
CS 170: Algorithms

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CS 170: Algorithms

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CS 170: Algorithms

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CS 170: Algorithms

Prims and Huffman Coding.
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CS 170: Algorithms

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CS 170: Algorithms

Prims and Huffman Coding.
Prims and Huffman Coding.
Lecture in a minute.

Cut Property: MST.
Exists MST that uses minimum weight edge across cut.
  Exchange argument. Prim: \( S = \{s\} \)
  Add cheapest edge \((u, v)\) across \((S, V - S)\)
  \( S = S + v \).
Repeat.
Use priority queue: \( O((|V| + |E|) \log |V|) \).
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Huffman Coding.
Symbols, $s$, with frequencies.
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Symbols, \( s \), with frequencies.
Prefix-Free code.
   \( \equiv \) binary tree with symbols at leaves.
Cost: sum of \( \text{depth}(s) \times \text{freq}(s) \).
Cost2: sum of frequencies of internal nodes.
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Huffman Coding.
Symbols, $s$, with frequencies.
Prefix-Free code.
   $\equiv$ binary tree with symbols at leaves.
Cost: sum of depth$(s) \times \text{freq}(s)$.
Cost2: sum of frequencies of internal nodes.
Algorithm: merge lowest frequency symbols, recurse.
   Exchange Argument $\implies$
   exists optimal tree with this structure.
What is a Cut?

What is a cut in an undirected graph, \( G = (V, E) \)?

(A) A set of edges whose removal disconnects a graph.

(B) For partition of \( V \), \((S, V - S)\), set of edges across it;
\[ E \cap (S \times V - S) \].

(A) and (B).

Note: sometimes specified as \((S, V - S)\)
..... sometimes explicitly as subset of edges \( E' \).
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What is the cut property for MSTs?

(A) Any edges in a cut is in some mst.
(B) The smallest edge in a cut is in some mst.
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![Diagram of a cut in a graph with sets S and V - S connected by edge e.](image)
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**Cut Property:** Any edge of minimal weight in a cut is in some MST. (If unique, it must be in MST.)
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**Cut Property:** Any edge of minimal weight in a cut is in some MST. (If unique, it must be in MST.)

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Proof: replace, $n - 1$ edges still connected and cheaper.
Prim’s algorithm

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Correctness: cut property with unique weight edges.

**Break ties for smallest edge according to lowest neighbors.**
Prim’s Algorithm.

**Generic MST:**

- Start with $S = v$.
- Find smallest edge $(x, y)$ in crossing $(S, V - S)$.
- Let $S = S \cup \{y\}$.

**Implementation?**

- Find smallest edge $(x, y)$?
- $O(m)$ time.
- $O(nm)$ time.

Look at same edges over and over again!

Use a priority queue to keep edges!

Actually...

Use a priority queue to keep “closest” node.
Prim's Algorithm.

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Prim(G,s)
foreach \( v \in V \): \( c(v) = \infty \), \( prev(v) = nil \)
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Prim’s Algorithm

**Prim(G,s)**

- **foreach** $v \in V$: $c(v) = \infty$, $prev(v) = nil$
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$H = \text{make}\_\text{pqueue}(V,c)$
Prim’s Algorithm

Prim(G,s)

\[ \begin{align*}
\text{foreach } v \in V: & \quad c(v) = \infty, \text{prev}(v) = \text{nil} \\
c(s) = 0, \text{prev}(s) = s \\
H = \text{make_pqueue}(V,c) \\
\text{while } (v = \text{delete_min}(H)):
\end{align*} \]
Prim’s Algorithm

\textbf{Prim}(G,s)

\begin{itemize}
  \item \textbf{foreach} \( v \in V: \ c(v) = \infty, \ \text{prev}(v) = nil \)
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  \item \( H = \text{make\_p\_queue}(V,c) \)
  \item \textbf{while} \ (v = \text{delete\_min}(H)):
    \begin{itemize}
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foreach $(v,w)$:

if ($w(v, w) < c(w)$):

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    foreach v ∈ V: c(v) = ∞, prev(v) = nil
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    while (v = deletemin(H)):
        foreach (v,w):
            if (w(v, w) < c(w)):
                c(w) = w(v, w), prev(w) = v
                decreaseKey(H,w)
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Dijkstra’s:

**Runtime?**

- \( \Theta(mn) \)
- \( \Theta((m+n)\log n) \)
- \( \Theta(m+n\log n) \)
- \( O((m+n)\log n) \)
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\hspace{1em} \textbf{if } c(v) + w(v, w) < c(w)):
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            \begin{itemize}
              \item \( c(w) = w(v,w), \text{prev}(w) = v \)
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            \end{itemize}
        \end{itemize}
    \end{itemize}
\end{itemize}

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\textbf{Runtime?} \( \Theta(mn) \)? \( \Theta((m+n)\log n) \)? \( \Theta(m+n\log n) \)? \( O((m+n)\log n) \)
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foreach (\( v,w \)):

if (\( w(v,w) < c(w) \)):

\( c(w) = w(v,w), \text{prev}(w) = v \)

decreaseKey(\( H,w \))

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Runtime? \( \Theta(mn) ? \Theta((m+n)\log n) ? \Theta(m+n\log n) ? O((m+n)\log n) \)

With Fibonacci Heaps: \( O(m+n\log n) \).
Compression.

Given a long file, make it shorter.
Compression.

Given a long file, make it shorter.

16 characters alphabet, four bits/character.

Morse code:

- E . “dot”
- T - “dash”
- I .. “dot dot”
- A .- “dot dash”
- N -. “dash dot”
- S ... “dot dot dot”

Common characters are shorter...

Cool!

Translate to 0 and 1?

Sure.

What is .. or “dot dot”?

“I” or “EE”??

Separate using pauses in morse code.. for binary?
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<thead>
<tr>
<th>Character</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
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<td>.</td>
</tr>
<tr>
<td>T</td>
<td>-</td>
</tr>
<tr>
<td>I</td>
<td>..</td>
</tr>
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<td>A</td>
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Cool! Translate to 0 and 1? Sure.

What is .. or “dot dot”?“|”
Compression.

Given a long file, make it shorter.

16 characters alphabet, four bits/character.
One Idea: shorter representation of frequent characters.

Morse code:

E . “dot”
T - “dash”
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Prefix free codes.

No code for a letter is a prefix of another.
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No code for a letter is a prefix of another.

Letters: A,B, C,D.
Prefix free codes.

No code for a letter is a prefix of another.

Letters: A, B, C, D.

Codewords: strings in \{0, 1\}^*.

Example:

A: 00
B: 01
C: 10
D: 11

Fixed length codewords. No codeword is prefix of another.

What is 100011?

First two: "C"
Next two: "A"
Third two: "D"
Can decode!
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"110010"
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CA B
DNA example.

Consists of letters $A, C, T, G$ with varying frequencies.
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$A$: .4  
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Yes!

Expected length: $N(.4*1 + .1*3 + .2 *3 + .3 * 2) = 1.9N$.

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A: .4  0
C: .1  100
T: .2
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\end{align*} \]

Expected length of fixed length encoding for \( N \) chars: \( 2N \).

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Prefix codes and trees.

Any prefix-free code corresponds to a full binary tree: each internal node has two children.
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```
A   0  1
  0  1
C   T
```

011101100
0 11 101 100
Prefix codes and trees.

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```
0
A
0
0
C
1
T
```

```
0 1 1 1 0 1 1 0 0
0 1 1 1 1 0 1 1 0 0
A
```
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Any prefix-free code corresponds to a full binary tree: each internal node has two children.
Tree from Prefix-Free Code.

Given prefix free code:
\[ S = \{ s_1, s_2, \ldots, s_n \} \] for symbols \( \{ c_1, \ldots, c_n \} \).
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Correctness: Every codeword/symbol corresponds to leaf.
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Correctness: Every codeword/symbol corresponds to leaf.
Let \( S_p \) be subset corresponding to node at "path" \( p \) in tree.
Corresponds to strings where \( p \) is prefix.
If there is internal node \( S_p \) with \( p \in S_p \).
\( p \) is prefix of another codeword.
Contradiction.
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Huffman Coding.

Given symbol frequencies $f_1, \ldots, f_n$, find “best” prefix code.
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Smallest average length.
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Smallest average length.
Cost of prefix tree with symbol leaves:

$$\sum_i f_i (\text{depth of symbol } i \text{ in tree.})$$
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Example: (A,.4), (C,.1),(T,.2),(G,.3)
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![Huffman Tree Diagram]

Cost: $0.4 \times 1$
Huffman Coding.

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Cost of prefix tree with symbol leaves:

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Example: $(A, .4), (C, .1), (T, .2), (G, .3)$

Cost: $0.4 \times 1 + 0.3 \times 2$
Huffman Coding.

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Example: \((A,.4), (C,.1), (T,.2), (G,.3)\)

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\begin{align*}
\text{Cost: } & .4 \times 1 + .3 \times 2 + .1 \times 3 \\
& = 1.9
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Example: \((A, .4), (C, .1), (T, .2), (G, .3)\)

Cost: \( .4 \times 1 + .3 \times 2 + .1 \times 3 + .2 \times 3 = 1.9 \)
Another view of cost.

Sum over all nodes, except root, of their frequency.
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[
\begin{align*}
(A,0.4) & + (C,0.1) + (T,0.2) + (G,0.3) \\
& = 0.4
\end{align*}
\]
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[ .4 + .1 = .9 \]

Optimal Tree should be optimal above any subtree.

E.g., Optimal tree on \{ (A, .4), (G, .3), (C, .1), (T, .2) \}. 
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[ .4 + .1 + .2 \]

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  (T, .2) \}.
Another view of cost.

\[ \sum_{\text{over all nodes, except root}} \text{of their frequency.} \]

\[ 0.4 + 0.1 + 0.2 + 0.3 = 1.0 \]

Optimal Tree should be optimal above any subtree. E.g., Optimal tree on \{ \(\cdot.4\), A \}, \{ \(\cdot.3\), \{ C, T \} \}, \{ \(\cdot.3\), G \} \}. 

Sum over all nodes, except root, of their frequency.

\[ .4 + .1 + .2 + .3 \]
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[.4 + .1 + .2 + .3 + .3\]
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[ .4 + .1 + .2 + .3 + .3 + .6 \]
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[ .4 + .1 + .2 + .3 + .3 + .6 = 1.9 \]
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[.4 + .1 + .2 + .3 + .3 + .6 = 1.9\]

Optimal Tree should be optimal above any subtree.
Another view of cost.

Sum over all nodes, except root, of their frequency.

\[ .4 + .1 + .2 + .3 + .3 + .6 = 1.9 \]

Optimal Tree should be optimal above any subtree.
E.g., Optimal tree on \{ (.4, A), (.3, \{ C, T \}), (.3, G) \}.
Greedy Algorithm.

Recursive View:
- An internal node has frequency $f_i$.
  - $f_i$ is a sort of symbol.
  - Idea: Merge two symbols to make a new internal node (symbol).
  - Which ones?
  - Cost of prefix tree with symbol leaves:
    $$\sum_i f_i \cdot \text{depth of symbol } i \text{ in tree.}$$
  - Might as well merge two lowest frequency symbols...
  - Let's see the method!

\begin{align*}
&A, 4, \{C, T\}, G, 3 \Rightarrow \\
&\{A, \{C, T\}, G\}, 6 \Rightarrow \\
&\{\{A, \{C, T\}\}, G\}, 1
\end{align*}
Greedy Algorithm.

Recursive View:
Greedy Algorithm.

Recursive View: internal node has frequency
Greedy Algorithm.

Recursive View: internal node has frequency
...”internal nodes” ≡ “sort of symbols.”
Greedy Algorithm.

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Which ones?
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Recursive View: internal node has frequency
...”internal nodes” ≡ “sort of symbols.”

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Which ones?

Cost of prefix tree with symbol leaves:
Greedy Algorithm.

Recursive View: internal node has frequency
..."internal nodes" ≡ "sort of symbols."

Idea: Merge two symbols to make new internal node (symbol).

Which ones?

Cost of prefix tree with symbol leaves:

\[ \sum_{i} f_i (\text{depth of symbol } i \text{ in tree.}) \]
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..."internal nodes" ≡ "sort of symbols."

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Might as well merge two lowest frequency symbols...
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Cost of prefix tree with symbol leaves:
\[ \sum_i f_i (\text{depth of symbol } i \text{ in tree.}) \]

Might as well merge two lowest frequency symbols...
to make low freq internal symbol.
Greedy Algorithm.

Recursive View: internal node has frequency
...”internal nodes” ≡ “sort of symbols.”

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Let's see method!

\((A, .4), (C, .1), (T, .2), (G, .3)\)
Greedy Algorithm.

Recursive View: internal node has frequency
...”internal nodes” ≡ “sort of symbols.”

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Let’s see method!

\((A, .4), (C, .1), (T, .2), (G, .3)\)
\((A, .4), \{C, T\}, .3), (G, .3)\)
Greedy Algorithm.

Recursive View: internal node has frequency
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Let’s see method!

\[(A, .4), (C, .1), (T, .2), (G, .3)\]
\[(A, .4), (\{C, T\}, .3), (G, .3)\]
\[(A, .4), (\{\{C, T\}, G\} .6)\]
Greedy Algorithm.

Recursive View: internal node has frequency
...”internal nodes”  \equiv “sort of symbols.”

Idea: Merge two symbols to make new internal node (symbol).
Which ones?

Cost of prefix tree with symbol leaves:

$$\sum_i f_i (\text{depth of symbol } i \text{ in tree.})$$

Might as well merge two lowest frequency symbols... to make low freq internal symbol.

Let’s see method!

$$(A, .4), (C, .1), (T, .2), (G, .3)$$

$$(A, .4), (\{C, T\}, .3), (G, .3)$$

$$(A, .4), (\{\{C, T\}, G\}.6)$$

$$\{A, \{\{C, T\}, G\}\}, 1)$$
Algorithm.

Cost2: Sum over all nodes, except root, of their “frequency”.

(A,.4)
(G,.3)
(C,.1)
(T,.2)

Implementation: priority queue to get lowest frequency trees.
Algorithm.

Cost2: Sum over all nodes, except root, of their “frequency”.

Algorithm:
Algorithm.

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Algorithm:
Make each symbol into single node tree.
Algorithm.

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While more than one tree:
Algorithm.

Cost2: Sum over all nodes, except root, of their “frequency”.

Algorithm:
Make each symbol into single node tree.
While more than one tree:
    Merge two lowest frequency trees, into a new tree.
Algorithm.

Cost2: Sum over all nodes, except root, of their “frequency”.

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(A,.4)

(G,.3)

(C,.1)    (T,.2)
Algorithm.

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Make each symbol into single node tree.
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(A,.4)

.3

(G,.3)

(C,.1) (T,.2)
Algorithm.

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Implementation: priority queue to get lowest frequency trees.
Correctness.

Recall MST: added a “could have” edge in tree every time.
Correctness..

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“must have” if no ties.
Correctness:

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Huffman algorithm: merge two lowest frequency symbols.
Correctness..
Recall MST: added a “could have” edge in tree every time. “must have” if no ties.

Huffman algorithm: merge two lowest frequency symbols. Make supersymbol.
Correctness.

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Correctness:
Exists: optimal tree where two lowest frequency symbols are siblings.
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**Correctness:**

 Exists: optimal tree where two lowest frequency symbols are siblings.
Correctness:
Recall MST: added a “could have” edge in tree every time.  
“must have” if no ties.

Huffman algorithm: merge two lowest frequency symbols.  
Make supersymbol.  Recurse.

Correctness:
Exists: optimal tree where two lowest frequency symbols are siblings.

Consider optimal tree.
Correctness

Recall MST: added a “could have” edge in tree every time.
“must have” if no ties.

Huffman algorithm: merge two lowest frequency symbols.
Make supersymbol. Recurse.

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Exists: optimal tree where two lowest frequency symbols are siblings.

Consider optimal tree.
Consider lowest frequency two symbols (assume no ties).
Correctness

Recall MST: added a “could have” edge in tree every time.
“must have” if no ties.

Huffman algorithm: merge two lowest frequency symbols.
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If siblings
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Correctness:
Exists: optimal tree where two lowest frequency symbols are siblings.

Consider optimal tree.
Consider lowest frequency two symbols (assume no ties).
If siblings → done.
Correctness:
Recall MST: added a “could have” edge in tree every time. 
“must have” if no ties.

Huffman algorithm: merge two lowest frequency symbols. 
Make supersymbol. Recurse.

**Correctness:**
Exists: optimal tree where two lowest frequency symbols are siblings.

Consider optimal tree.
Consider lowest frequency two symbols (assume no ties). 
If siblings $\rightarrow$ done.
Otherwise ... switch each with deepest pair of siblings improves tree.
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Recall MST: added a “could have” edge in tree every time.
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Consider optimal tree. Consider lowest frequency two symbols (assume no ties). If siblings $\rightarrow$ done. Otherwise ... switch each with deepest pair of siblings improves tree.

Cost gets better with this switch.
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Correctness:
Exists: optimal tree where two lowest frequency symbols are siblings.

Consider optimal tree.
Consider lowest frequency two symbols (assume no ties).
If siblings $\rightarrow$ done.
Otherwise ... switch each with deepest pair of siblings improves tree.

Cost gets better with this switch. Lowest frequency pair are now siblings!
Lowest frequency at same depth.

What about same depth?

\[
\begin{align*}
\text{(y, 0.01)} & \quad \text{(z, 0.04)} \\
\text{(x, 0.1)}
\end{align*}
\]
Lowest frequency at same depth.

What about same depth?

```
        .
       .
      .
     .
    .
   (x,.1)  (z,.04)  (y,.01)
```

Cost stays the same,
Lowest frequency at same depth.

What about same depth?

Cost stays the same, but lowest pair are siblings.
Lowest frequency at same depth.

What about same depth?

Cost stays the same, but lowest pair are siblings.

⇒ There is optimal tree where lowest frequency pair are siblings.
Lowest frequency at same depth.

What about same depth?

Cost stays the same, but lowest pair are siblings.

\[ \Rightarrow \text{There is optimal tree where lowest frequency pair are siblings.} \]

“Algorithm: merge lowest frequency pair, and recurse.”
Lowest frequency at same depth.

What about same depth?

Cost stays the same, but lowest pair are siblings.

⇒ There is optimal tree where lowest frequency pair are siblings.

“Algorithm: merge lowest frequency pair, and recurse.”

Produces optimal tree.
Cut Property: MST.
Exists MST that uses minimum weight edge across cut.
  Exchange argument. Prim: $S = \{s\}$
   Add cheapest edge $(u, v)$ across $(S, V - S)$
   $S = S + v$.
Repeat.
Use priority queue: $O((|V| + |E|) \log |V|)$. 

Huffman Coding.
Symbols, $s$, with frequencies.
Prefix-Free code. ≡ binary tree with symbols at leaves.
Cost: sum of depth(s) $\times$ freq($s$).
Cost2: sum of frequencies of internal nodes.
Algorithm: merge lowest frequency symbols, recurse.
Exchange Argument $\Rightarrow$ exists optimal tree with this structure.
Lecture in a minute.

Cut Property: MST.
 Exists MST that uses minimum weight edge across cut.
   Exchange argument. Prim: \( S = \{s\} \)
   Add cheapest edge \((u, v)\) across \((S, V - S)\)
   \( S = S + v. \)
Repeat.
Use priority queue: \( O((|V| + |E|) \log |V|). \)

Huffman Coding.
Symbols, \( s \), with frequencies.
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Exists MST that uses minimum weight edge across cut.
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Prefix-Free code.
  \( \equiv \) binary tree with symbols at leaves.
Cost: sum of depth\((s) \times freq\((s)\).
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