CS170: Algorithms
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Lecture in a minute.

Horn Formula:
- Implications of positive literals with ANDs on one side.
- Plus ORs of negatives.
- Negative clauses problem only with true literals.
- Greedy: only set true if have to.
Lecture in a minute.

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Set Cover:
- Given subsets of some elements.
- Find: min number of sets that contains every element.
- Greedy: choose largest set w.r.t. remaining elements.
- $O(\log n)$ approximate solution.
- Proof Idea: optimal of size $k \implies$ Cover $1/k$ of the remaining elts.
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Set Cover:
Given subsets of some elements.
Find: min number of sets that contains every element.
Greedy: choose largest set w.r.t. remaining elements.
$O(\log n)$ approximate solution.
Proof Idea: optimal of size $k \Rightarrow$ Cover $1/k$ of the remaining elts.

Path Compression:
$O(m \log^* n)$ time for $m$ finds.
Some finds expensive but cheap on average.
Idea: group ranks into $\log^* n$ sets.
Small number of pointers across sets in any find.
Total movement inside sets $O(n)$.
Idea: from not more than $2^k$ nodes of rank $k$. 
Taint Analysis

```java
Taint Analysis

a = http.read(response);
b = a + c;
d = sql.command(b);
a is input from web. "a is tainted."
"if a is tainted b is tainted."
"b should not be tainted."
Logic representation:
A - "a is tainted"
B - "b is tainted"
A =⇒ A, A =⇒ B, B.
Satisfiable?
Not in this case.
```
Taint Analysis

\[ a = \text{http.read_response}(); \]

"a is tainted."

"if a is tainted, b is tainted."

"b should not be tainted."

Logic representation:

- \( A \) - "a is tainted"
- \( B \) - "b is tainted"

\[ A \implies B, A = \neg B. \]

Satisfiable?

Not in this case.
Taint Analysis

\[ a = \text{http.read_response}() \]

\[ b = a + c; \]

\[ d = \text{sql.command}(b); \]

*a* is input from web.

"*a* is tainted."

"if *a* is tainted, *b* is tainted."

"*b* should not be tainted."

Logic representation:

\[ A \iff \text{not } A, \ A = \Rightarrow \ B, \ B = \text{not } B. \]

Satisfiable? Not in this case.
Taint Analysis

\[ a = \text{http.read.response}(); \]

\[ b = a + c; \]

"a is input from web. "

"a is tainted."

"if a is tainted, b is tainted."

"b should not be tainted."

Logic representation:

\[ A \implies \neg A, \quad A \implies \neg B, \quad B. \]

Satisfiable?

Not in this case.
Taint Analysis

\[
a = \text{http.read_response}(); \\
\vdots \\
b = a + c; \\
\vdots \\
d = \text{sql\_command}(b);
\]

"a is input from web. "
"a is tainted."
"if a is tainted b is tainted."
"b should not be tainted."

Logic representation:

\[
A \implies A, \quad A \implies B 
\]

Satisfiable? Not in this case.
Taint Analysis

\[ a = \text{http.read_response}(); \]
\[ b = a + c; \]
\[ d = \text{sql.command}(b); \]

\( a \) is input from web.

"\( a \) is tainted."

"if \( a \) is tainted \( b \) is tainted."

"\( b \) should not be tainted."

Logic representation:

\[ A - \text{"a is tainted"}, \quad A = \Rightarrow B, \quad B. \]

Satisfiable? Not in this case.
Taint Analysis

\[ a = \text{http.read_response}(); \]
\[ b = a + c; \]
\[ d = \text{sql\_command}(b); \]

*a is input from web.*

“*a is tainted.”*
Taint Analysis

\[ a = \text{http.read_response}(); \]
\[ b = a + c; \]
\[ d = \text{sql.command}(b); \]

\( a \) is input from web.
“\( a \) is tainted.”
“if \( a \) is tainted \( b \) is tainted.”
Taint Analysis

\[ a = \text{http.read_response}(); \]
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“\( a \) is tainted.”
“if \( a \) is tainted \( b \) is tainted.”
“\( b \) should not be tainted.”

Logic representation:

\[ A \iff A, B \iff \neg B. \]

Satisfiable?
Not in this case.
Taint Analysis

\[ a = \text{http.read_response}(); \]
\[ b = a + c; \]
\[ d = \text{sql_command}(b); \]

\( a \) is input from web.
“\( a \) is tainted.”
“if \( a \) is tainted \( b \) is tainted.”
“\( b \) should not be tainted.”

Logic representation:
\( A - \text{“a is tainted”} \)
Taint Analysis

\[
a = \text{http.read_response}()
\]
\[
\vdots
\]
\[
b = a + c;
\]
\[
\vdots
\]
\[
d = \text{sql.command}(b);
\]

\(a\) is input from web.
“\(a\) is tainted.”
“if \(a\) is tainted \(b\) is tainted.”
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Logic representation:
\(A\) - “\(a\) is tainted”
\(B\) - “\(b\) is tainted”
Taint Analysis

\[ a = \text{http.read_response}(); \]
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\( a \) is input from web.
“\( a \) is tainted.”
“if \( a \) is tainted \( b \) is tainted.”
“\( b \) should not be tainted.”

Logic representation:
\( A - \text{“a is tainted”} \)
\( B - \text{“b is tainted”} \)

\[ \rightarrow A \]
Taint Analysis

\[ a = \text{http.read_response}(); \]

\[ b = a + c; \]

\[ d = \text{sql.command}(b); \]

\( a \) is input from web.
"\( a \) is tainted."
"if \( a \) is tainted \( b \) is tainted."
"\( b \) should not be tainted."

Logic representation:
\[ A - \text{“a is tainted”} \]
\[ B - \text{“b is tainted”} \]

\[ \implies A, A \implies B \]
Taint Analysis

\[ a = \text{http.read_response}(); \]
\[ b = a + c; \]
\[ d = \text{sql\_command}(b); \]

\( a \) is input from web.
“\( a \) is tainted.”
“if \( a \) is tainted \( b \) is tainted.”
“\( b \) should not be tainted.”

Logic representation:
\( A - \text{“a is tainted”} \)
\( B - \text{“b is tainted”} \)

\( \rightarrow A, A \rightarrow B, \bar{B}. \)
Taint Analysis

\[ a = \text{http.read\_response}() \]

\[ b = a + c \]

\[ d = \text{sql\_command}(b) \]

\( a \) is input from web.

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\( A - \text{“a is tainted”} \)

\( B - \text{“b is tainted”} \)

\[ \implies A, A \implies B, \overline{B} \]

Satisfiable?
Taint Analysis

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Logic representation:
\( A \) - “\( a \) is tainted”
\( B \) - “\( b \) is tainted”

\[ \rightarrow A , A \rightarrow B , \overline{B}. \]

Satisfiable?
Not in this case.
Horn SAT

Implications:

$x \land y \rightarrow z$

Negative clauses:

$u \lor v$

Taint Example:

$= \Rightarrow A$, $A = \Rightarrow B$, $B$. Is this satisfiable? True is the problem. If every literal is False:

All $\land$ implication statements are good.
All $\lor$ statements are true.
except for implication: $= \Rightarrow A$. This forces a true literal.
Horn SAT

Implications:

And of positive literals imply one positive literal.
Horn SAT

Implications:

And of positive literals imply one positive literal.

\[ x \land y \rightarrow z \]
Horn SAT

Implications:

**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:
Horn SAT

Implications:
	**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:
	**Or** of negative literals.

Taint Example:
\[ = \Rightarrow A, \quad A \Rightarrow B \]

Is this satisfiable?
True is the problem.

If every literal is False:
All \( \land \) implication statements are good.
All \( \lor \) statements are true.
except for implication:
\[ = \Rightarrow A \].

This forces a true literal.
Horn SAT

Implications:

**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:

**Or** of negative literals.

\[ \overline{u} \lor \overline{v} \]
Horn SAT

Implications:
- **And** of positive literals imply **one** positive literal.
  \[ x \land y \rightarrow z \]

Negative clauses:
- **Or** of negative literals.
  \[ \bar{u} \lor \bar{v} \]

Taint Example:
Implications:
   **And** of positive literals imply **one** positive literal.
   \[ x \land y \rightarrow z \]

Negative clauses:
   **Or** of negative literals.
   \[ \overline{u} \lor \overline{v} \]

Taint Example:
   \[ \Rightarrow A, \overline{A} \Rightarrow B, \overline{B}. \]
Horn SAT

Implications:

**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:

**Or** of negative literals.

\[ \overline{u} \lor \overline{v} \]

Taint Example:

\[ \Rightarrow A, A \Rightarrow B, \overline{B} \]

Is this satisfiable?
Horn SAT

Implications:

And of positive literals imply one positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:

Or of negative literals.

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Taint Example:

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Negative clauses:

Or of negative literals.

\[ \bar{u} \lor \bar{v} \]

Taint Example:

\[ \Rightarrow A, A \Rightarrow B, \bar{B} \]

Is this satisfiable?

True is the problem.

If every literal is False:


Horn SAT

Implications:

**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:

**Or** of negative literals.

\[ \overline{u} \lor \overline{v} \]

Taint Example:

\[ \rightarrow A, \quad A \rightarrow B, \overline{B}. \]

Is this satisfiable?

**True** is the problem.

If every literal is **False**:

All \( \land \) implication statements are good.
Implications:

**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:

**Or** of negative literals.

\[ \overline{u} \lor \overline{v} \]

Taint Example:

\[ \Rightarrow A, \ A \Rightarrow B, \overline{B}. \]

Is this satisfiable?

**True** is the problem.

If every literal is **False**:

- All \( \land \) implication statements are good.
- All \( \lor \) statements are true.
Horn SAT

Implications:

**And** of positive literals imply **one** positive literal.

\[ x \land y \rightarrow z \]

Negative clauses:

**Or** of negative literals.

\[ \bar{u} \lor \bar{v} \]

Taint Example:

\[ \implies A, A \implies B, \bar{B}. \]

Is this satisfiable?

**True** is the problem.

If every literal is **False**:

- All \( \land \) implication statements are good.
- All \( \lor \) statements are true.

except for implication: \( \implies A. \)
Horn SAT

Implications:
   **And** of positive literals imply **one** positive literal.
   \[ x \land y \rightarrow z \]

Negative clauses:
   **Or** of negative literals.
   \[ \overline{u} \lor \overline{v} \]

Taint Example:
   \[ \implies A, A \implies B, \overline{B} \].

Is this satisfiable?

**True** is the problem.
   If every literal is **False**:
      All \( \land \) implication statements are good.
      All \( \lor \) statements are true.
      except for implication: \( \implies A \).

This forces a true literal.
Horn Sat: another view.

\[
x_1 \land x_2 \implies x_4 \\
x_3 \implies x_2 \\
x_1 \implies x_3 \\
x_5 \land x_1 \implies x_3 \\
x_2 \land x_6 \implies x_5 \\
\implies x_1
\]
Horn Sat: another view.

\[ x_1 \land x_2 \implies x_4 \]
\[ x_3 \implies x_2 \]
\[ x_1 \implies x_3 \]
\[ x_5 \land x_1 \implies x_3 \]
\[ x_2 \land x_6 \implies x_5 \]
\[ \implies x_1 \]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm:
Only set literals to True if you have to.

Example:
- \( x_1 \) must be True
- \( x_3 \) must be True
- \( x_2 \) must be True
- \( x_4 \) must be True

Solution:
\{ \( x_1 \), \( x_2 \), \( x_3 \), \( x_4 \) \} are True

Could also set \( x_5 \) to true, or both \( x_5 \) and \( x_6 \) to true...

but don't!

Same as horn sat!
Horn Sat: another view.

\[ x_1 \land x_2 \implies x_4 \]
\[ x_3 \implies x_2 \]
\[ x_1 \implies x_3 \]
\[ x_5 \land x_1 \implies x_3 \]
\[ x_2 \land x_6 \implies x_5 \]
\[ \implies x_1 \]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm:
Horn Sat: another view.

\[ x_1 \land x_2 \implies x_4 \]
\[ x_3 \implies x_2 \]
\[ x_1 \implies x_3 \]
\[ x_5 \land x_1 \implies x_3 \]
\[ x_2 \land x_6 \implies x_5 \]
\[ \implies x_1 \]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.
Horn Sat: another view.

\begin{align*}
  x_1 \land x_2 & \implies x_4 \\
  x_3 & \implies x_2 \\
  x_1 & \implies x_3 \\
  x_5 \land x_1 & \implies x_3 \\
  x_2 \land x_6 & \implies x_5 \\
  & \implies x_1 
\end{align*}

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
x_1 must be True
Horn Sat: another view.

\[
\begin{align*}
    x_1 \land x_2 & \implies x_4 \\
    x_3 & \implies x_2 \\
    x_1 & \implies x_3 \\
    x_5 \land x_1 & \implies x_3 \\
    x_2 \land x_6 & \implies x_5 \\
    & \implies x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to \textbf{True} if you have to.

Example:
\( x_1 \) must be \textbf{True}
Horn Sat: another view.

\[ x_1 \land x_2 \implies x_4 \]
\[ x_3 \implies x_2 \]
\[ x_1 \implies x_3 \]
\[ x_5 \land x_1 \implies x_3 \]
\[ x_2 \land x_6 \implies x_5 \]
\[ \implies x_1 \]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
\[ x_1 \] must be True so \[ x_3 \] must be True
Horn Sat: another view.

\[
\begin{align*}
  x_1 \land x_2 & \implies x_4 \\
  x_3 & \implies x_2 \\
  x_1 & \implies x_3 \\
  x_5 \land x_1 & \implies x_3 \\
  x_2 \land x_6 & \implies x_5 \\
  & \implies x_1 
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:

\(x_1\) must be True so \(x_3\) must be True
Horn Sat: another view.

\[
\begin{align*}
x_1 \land x_2 & \quad \implies \quad x_4 \\
x_3 & \quad \implies \quad x_2 \\
x_1 & \quad \implies \quad x_3 \\
x_5 \land x_1 & \quad \implies \quad x_3 \\
x_2 \land x_6 & \quad \implies \quad x_5 \\
& \quad \implies \quad x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
- \(x_1\) must be True so \(x_3\) must be True
- so \(x_2\) must be True

Solution:
\{\(x_1\), \(x_2\), \(x_3\), \(x_4\)\} are True

Could also set \(x_5\) to true, or both \(x_5\) and \(x_6\) to true...

but don’t! Same as horn sat!
Horn Sat: another view.

\[
\begin{align*}
x_1 \land x_2 & \implies x_4 \\
x_3 & \implies x_2 \\
x_1 & \implies x_3 \\
x_5 \land x_1 & \implies x_3 \\
x_2 \land x_6 & \implies x_5 \\
& \implies x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
\(x_1\) must be True so \(x_3\) must be True
so \(x_2\) must be True
Horn Sat: another view.

\[
\begin{align*}
x_1 \land x_2 & \implies x_4 \\
x_3 & \implies x_2 \\
x_1 & \implies x_3 \\
x_5 \land x_1 & \implies x_3 \\
x_2 \land x_6 & \implies x_5 \\
& \implies x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
\(x_1\) must be True so \(x_3\) must be True
so \(x_2\) must be True so \(x_4\) must be True
Horn Sat: another view.

\[ x_1 \land x_2 \implies x_4 \]
\[ x_3 \implies x_2 \]
\[ x_1 \implies x_3 \]
\[ x_5 \land x_1 \implies x_3 \]
\[ x_2 \land x_6 \implies x_5 \]
\[ \implies x_1 \]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
\( x_1 \) must be True so \( x_3 \) must be True
so \( x_2 \) must be True so \( x_4 \) must be True

Solution:
Horn Sat: another view.

\[
\begin{align*}
    x_1 \land x_2 & \implies x_4 \\
    x_3 & \implies x_2 \\
    x_1 & \implies x_3 \\
    x_5 \land x_1 & \implies x_3 \\
    x_2 \land x_6 & \implies x_5 \\
    & \implies x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
\begin{align*}
    x_1 \text{ must be } \text{True} \text{ so } x_3 \text{ must be } \text{True} \\
    \text{so } x_2 \text{ must be } \text{True} \text{ so } x_4 \text{ must be } \text{True}
\end{align*}

Solution: \{x_1, x_2, x_3, x_4\} are True
Horn Sat: another view.

\[
x_1 \land x_2 \implies x_4
\]

\[
x_3 \implies x_2
\]

\[
x_1 \implies x_3
\]

\[
x_5 \land x_1 \implies x_3
\]

\[
x_2 \land x_6 \implies x_5
\]

\[
\implies x_1
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:

- \( x_1 \) must be True so \( x_3 \) must be True
- so \( x_2 \) must be True so \( x_4 \) must be True

Solution: \( \{ x_1, x_2, x_3, x_4 \} \) are True

Could also set \( x_5 \) to true, or both \( x_5 \) and \( x_6 \) to true...
Horn Sat: another view.

\[
\begin{align*}
& x_1 \land x_2 \implies x_4 \\
& x_3 \implies x_2 \\
& x_1 \implies x_3 \\
& x_5 \land x_1 \implies x_3 \\
& x_2 \land x_6 \implies x_5 \\
& \implies x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to \textit{True} if you have to.

Example:
\(x_1\) must be \textit{True} so \(x_3\) must be \textit{True}
so \(x_2\) must be \textit{True} so \(x_4\) must be \textit{True}

Solution: \{\(x_1, x_2, x_3, x_4\}\} are \textit{True}

Could also set \(x_5\) to true, or both \(x_5\) and \(x_6\) to true...but don’t!
Horn Sat: another view.

\[
\begin{align*}
  x_1 \land x_2 & \implies x_4 \\
  x_3 & \implies x_2 \\
  x_1 & \implies x_3 \\
  x_5 \land x_1 & \implies x_3 \\
  x_2 \land x_6 & \implies x_5 \\
  & \implies x_1
\end{align*}
\]

Problem: Find consistent assignment with fewest “True” literals.

Greedy algorithm: Only set literals to True if you have to.

Example:
\begin{itemize}
  \item $x_1$ must be True so $x_3$ must be True
  \item so $x_2$ must be True so $x_4$ must be True
\end{itemize}

Solution: \{x_1, x_2, x_3, x_4\} are True

Could also set $x_5$ to true, or both $x_5$ and $x_6$ to true...but don’t!

Same as horn sat!
Why same as HornSAT?

Horn SAT had negative clauses.
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
Property: any variable set to true must be true in any satisfying assignment.
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
Property: any variable set to true must be true in any satisfying assignment.
By induction.
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
Property: any variable set to true must be true in any satisfying assignment.
By induction. First $k$ set to true...
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
Property: any variable set to true must be true in any satisfying assignment.
By induction. First $k$ set to true... must be!
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
Property: any variable set to true must be true in any satisfying assignment.
By induction. First $k$ set to true... must be!
The $k+1$ set variable set to true
Why same as HornSAT?

Horn SAT had negative clauses.
No negative clauses for above algorithm.
Algorithm: Set a variable true ..if you have to!
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So if a negative clause is false, it must be.
Any SAT formula?

$x_1 \iff x_2 \lor x_3$. 

$x_1$ being true may mean nothing for $x_3$? 

No known polynomial time algorithm.

...no polynomial time algorithm unless NP = P ...

"P = NP?"

"There is an efficient algorithm to find a solution for every problem whose solution can be efficiently checked."

More later...in the course.
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“\( \text{P} = \text{NP} \)”?

“There is an efficient algorithm to \textbf{find} a solution for every problem whose solution can be efficiently \textbf{checked}.”

More later...in the course.
Set Cover.

Input:

Items: City Blocks.
Sets: Possible cellphone tower location. Each cell phone tower location covers some subset of blocks.

Items: Customers.
Sets: Walmart locations covers subset of customers.

Items: Job responsibilities (ruby, perl, python, web, unix,...).
Sets: People with job capabilities.

Items: factory needs (touch screens, chips, cheap labor).
Sets: suppliers. (Thousands of suppliers for GM!!)
Set Cover.

Input:
Items: $B = \{1, \ldots, n\}$
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Sets: $S_1, \ldots, S_m \subseteq B$
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Find fewest sets that cover $B$ (so that union is $B$)
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Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
Remove elements in $S_i$ from all sets
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
Remove elements in $S_i$ from all sets (Rinse).

When do we stop?
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
Remove elements in $S_i$ from all sets (Rinse).
Repeat.
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
Remove elements in $S_i$ from all sets (Rinse).
Repeat.

Number of sets is number of iterations.
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
   Remove elements in $S_i$ from all sets (Rinse).
   Repeat.

Number of sets is number of iterations.
How many iterations?
Choose set $S_i$ that has largest number of elts.
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How many iterations?

Property: Set cover of size $k$
Greedy Algorithm

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   Remove elements in $S_i$ from all sets (Rinse).
   Repeat.

Number of sets is number of iterations.
How many iterations?

Property: Set cover of size $k$ (best solution)
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
   Remove elements in $S_i$ from all sets (Rinse).
   Repeat.

Number of sets is number of iterations.
How many iterations?

Property: Set cover of size $k$ (best solution)
   $\implies$ a set contains $\frac{1}{k}$ of remaining elements.
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
  Remove elements in $S_i$ from all sets (Rinse).
  Repeat.

Number of sets is number of iterations.
How many iterations?

Property: Set cover of size $k$ (best solution)
  $\implies$ a set contains $\frac{1}{k}$ of remaining elements.

Analysis:
$n_t$ elements remain at time $t$ (after using $t$ sets.)
Choose set $S_i$ that has largest number of elts.
   Remove elements in $S_i$ from all sets (Rinse).
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Number of sets is number of iterations.
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$n_t$ elements remain at time $t$ (after using $t$ sets.)
In iteration $t$, cover $\frac{1}{k} n_t$ remaining elements.
Greedy Algorithm

Choose set $S_i$ that has largest number of elts.
   Remove elements in $S_i$ from all sets (Rinse).
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\[ n_{t+1} \leq \]
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$$n_{t+1} \leq n_t - \frac{1}{k} n_t$$
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$$n_{t+1} \leq n_t - \frac{1}{k} n_t = (1 - \frac{1}{k}) n_t.$$
Greedy Algorithm

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Repeat.

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$$n_t \leq (1 - \frac{1}{k})^t n_0$$
Greedy Algorithm

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When do we stop?
Bound iterations.

When do we stop?
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When do we stop?
When $n_t < 1$?
Bound iterations.

When do we stop?
When \( n_t < 1 \)?
Recall: \( n_t \leq (1 - \frac{1}{k})^t n_0 \)
Bound iterations.

When do we stop?

When $n_t < 1$?

Recall: $n_t \leq (1 - \frac{1}{k})^t n_0$

For what $t$ must this be true?
When do we stop?
When \( n_t < 1 \)?
Recall: \( n_t \leq (1 - \frac{1}{k})^t n_0 \)
For what \( t \) must this be true?

(A) \( t = \log n \)
(B) \( t = k \)
(C) \( t = k \log n \).
When do we stop?
When \( n_t < 1 \)?
Recall: \( n_t \leq (1 - \frac{1}{k})^t n_0 \)
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(C).
Plug in $t = k \ln n$ and $n_t < 1$. 
Bound iterations.

When do we stop?
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Recall: \( n_t \leq (1 - \frac{1}{k})^t n_0 \)

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(C).

Plug in \( t = k \ln n \) and \( n_t < 1 \).

In more detail...
Bound iterations (really)

\[ n_t \leq (1 - \frac{1}{k})^t n_0 \]
Bound iterations (really)

\[ n_t \leq (1 - \frac{1}{k})^t n_0 \]
When must \( n_t < 1? \)
Bound iterations (really)

\[ n_t \leq (1 - \frac{1}{k})^t n_0 \]

When must \( n_t < 1 \)?

Of course you remember:

\[ f(x) = e^{-x} \]

So, \( n_t \leq (1 - 1/k)^t n_0 < (e^{-1/k})^t n_0 \leq (e^{-t/k}) n_0 \).

For \( t = k \ln n \),

\[ n_t < (e^{-\ln n}) n = (\frac{1}{n}) n = 1. \]

No elements are uncovered at this time!

So \( t \leq k \ln n \).

Number of sets for greedy is at most \( k \ln n \)!

Within \( \ln n \) of \( k \), which is the best possible!
Bound iterations (really)

\[ n_t \leq (1 - \frac{1}{k})^t n_0 \]

When must \( n_t < 1? \)

Of course you remember: \( (1 - x) \leq e^{-x} \)
Bound iterations (really)

\[ n_t \leq (1 - \frac{1}{k})^t n_0 \]

When must \( n_t < 1 \)?

Of course you remember: \((1 - x) \leq e^{-x}\) Smile!
Bound iterations (really)

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Bound iterations (really)

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When must \( n_t < 1? \)

Of course you remember: \( (1 - x) \leq e^{-x} \) Smile!

So, \( n_t \leq (1 - \frac{1}{k})^t n \)
Bound iterations (really)

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When must \( n_t < 1 \)?

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\[ f(x) = e^{-x} \]

\[ 1 - x \]

So, \( n_t \leq (1 - \frac{1}{k})^t n < (e^{-\frac{1}{k}})^t n \)
Bound iterations (really)

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No elements are uncovered at this time!
Bound iterations (really)

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No elements are uncovered at this time!

So \( t \leq k \ln n \). Number of sets for greedy is at most \( k \ln n \)!
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So \( t \leq k \ln n \). Number of sets for greedy is at most \( k \ln n \! \)

Within \( \ln n \) of \( k \),
Bound iterations (really)

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Within \( \ln n \) of \( k \), which is the best possible!
Hmmm...

We did not find optimal solution!
Hmmm...

We did not find optimal solution!
Is there a better analysis?
Hmmm...

We did not find optimal solution!
Is there a better analysis?
No.
Hmmm...

We did not find optimal solution!
Is there a better analysis?
No. Problem 5.33!
Hmmm...

We did not find optimal solution!
Is there a better analysis?
No. Problem 5.33!
   Idea: Two sets cover all the elements.
We did not find optimal solution!
Is there a better analysis?
No. Problem 5.33!

Idea: Two sets cover all the elements.
One set covers slightly more than half the remaining elements.
We did not find optimal solution!

Is there a better analysis?

No. Problem 5.33!

Idea: Two sets cover all the elements.
    One set covers slightly more than half the remaining elements.
    Give $\Omega(\log n)$ lower bound.
We did not find optimal solution!

Is there a better analysis?

No. Problem 5.33!

Idea: Two sets cover all the elements.
One set covers slightly more than half the remaining elements.
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Is there a better algorithm?
We did not find optimal solution!
Is there a better analysis?
No. Problem 5.33!

Idea: Two sets cover all the elements.
One set covers slightly more than half the remaining elements.
Give $\Omega(\log n)$ lower bound.

Is there a better algorithm?
“Probably” not!
Hmmm...

We did not find optimal solution!
Is there a better analysis?
No. Problem 5.33!

Idea: Two sets cover all the elements.
One set covers slightly more than half the remaining elements.
Give $\Omega(\log n)$ lower bound.

Is there a better algorithm?
“Probably” not!
Again, only if P=NP.
We did not find optimal solution!

Is there a better analysis?

No. Problem 5.33!

   Idea: Two sets cover all the elements.
      One set covers slightly more than half the remaining elements.
      Give $\Omega(\log n)$ lower bound.

Is there a better algorithm?

“Probably” not!

Again, only if $P=NP$.

More later in the course.
Disjoint Set Data Structure

Maintain pointers: \( \pi(x) \) for each \( x \). Initially: \( \text{rank}(x) = 0 \).
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

**makeset(x)**
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

\textbf{makeset}(x) $\pi(x) = x$. 
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

**makeset**($x$) $\pi(x) = x$.

**find**($x$)
  - if $\pi(x) == x$
    - return $x$
  - else
    - find($\pi(x)$)

**union**($x$, $y$)
  - $r_x = \text{find}(x)$
  - $r_y = \text{find}(y)$
  - if $\text{rank}(r_x) < \text{rank}(r_y)$:
    - $\pi(r_x) = r_y$
  - else:
    - $\pi(r_y) = r_x$
  - if $\text{rank}(r_x) == \text{rank}(r_y)$:
    - $\text{rank}(r_x) += 1$

Properties:
1. Parent has a strictly higher rank.
2. Rank doesn't change for internal nodes.
3. $\text{rank}(x) = \text{rank}(y) = k$
   - (i) Each have $\geq 2^k$ vertices in sets
   - (ii) and the sets are disjoint.
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

- **makeset(x)** $\pi(x) = x$.
- **find(x)**
  - if $\pi(x) == x$
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- **union(x,y)**
  - $r_x = \text{find}(x)$
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Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

\begin{align*}
\text{makeset}(x) & \quad \pi(x) = x. \\
\text{find}(x) & \quad \text{if } \pi(x) == x \quad \text{return } x \\
& \quad \text{else} \\
& \quad \quad \text{find}(\pi(x)) \\
\end{align*}

\begin{align*}
\text{union}(x,y) & \\
& \quad r_x = \text{find}(x) \\
& \quad r_y = \text{find}(y) \\
& \quad \text{if } \text{rank}(r_x) < \text{rank}(r_y): \\
& \quad \quad \pi(r_x) = r_y \\
& \quad \text{else:} \\
& \quad \quad \pi(r_y) = r_x \\
& \quad \quad \text{if } \text{rank}(r_x) == \text{rank}(r_y): \\
& \quad \quad \quad \text{rank}(r_x) += 1
\end{align*}

Properties:

(1) Parent has a strictly higher rank.
(2) Rank doesn't change for internal nodes.
(3) $\text{rank}(x) = \text{rank}(y) = k$
   (i) Each have $\geq 2^k$ vertices in sets
   (ii) and the sets are disjoint.
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

**makeSet(x)** $\pi(x) = x$.

**find(x)**
  - if $\pi(x) == x$
    - return $x$
  - else
    - find($\pi(x)$)

**union(x,y)**
  - $r_x = \text{find}(x)$
  - $r_y = \text{find}(y)$
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  - else:
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Properties:
(1) Parent has a strictly higher rank.
Disjoint Set Data Structure

Maintain pointers: $\pi(x)$ for each $x$. Initially: $\text{rank}(x) = 0$.

```plaintext
makeSet(x) $\pi(x) = x$.

find(x)
  if $\pi(x) == x$
    return x
  else
    find($\pi(x)$)

union(x, y)
  $r_x = \text{find}(x)$
  $r_y = \text{find}(y)$
  if $\text{rank}(r_x) < \text{rank}(r_y)$:
    $\pi(r_x) = r_y$
  else:
    $\pi(r_y) = r_x$
    if $\text{rank}(r_x) == \text{rank}(r_y)$:
      $\text{rank}(r_x) + = 1$
```

Properties:
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1. Parent has a strictly higher rank.
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**makeset(x)**  $\pi(x) = x$.

**find(x)**
if $\pi(x) == x$
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**union(x,y)**

$r_x = \text{find}(x)$
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Properties:
(1) Parent has a strictly higher rank.
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Disjoint Set Data Structure

Maintain pointers: \( \pi(x) \) for each \( x \). Initially: \( \text{rank}(x) = 0 \).

**makeset**\( (x) \) \( \pi(x) = x \).

**find**\( (x) \)
  
  if \( \pi(x) == x \)
    return \( x \)
  else
    find(\( \pi(x) \))

**union**\( (x,y) \)
  
  \( r_x = \text{find}(x) \)
  \( r_y = \text{find}(y) \)
  
  if \( \text{rank}(r_x) < \text{rank}(r_y) \):
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  else:
    \( \pi(r_y) = r_x \)
    if \( \text{rank}(r_x) == \text{rank}(r_y) \):
      \( \text{rank}(r_x) += 1 \)

Properties:
(1) Parent has a strictly higher rank.
(2) Rank doesn’t change for internal nodes.
(3) \( \text{rank}(x) = \text{rank}(y) = k \)
  (i) Each have \( \geq 2^k \) vertices in sets
  (ii) and the sets are disjoint.
Path Compression

```python
def find(x):
    if π(x) == x:
        return x
    else:
        return find(π(x))
```

What happens if we find(x) again?
Chase again!
Path Compression

\[
\text{find}(x)
\begin{align*}
\text{if } \pi(x) &= x \\
\text{return } x \\
\text{else} \\
\text{find}(\pi(x))
\end{align*}
\]

What happens if we find\((x)\) again?
Path Compression

\[
\text{find}(x) \\
\quad \text{if } \pi(x) == x \\
\quad \text{return } x \\
\quad \text{else} \\
\quad \quad \text{find}(\pi(x))
\]

What happens if we find(x) again?
Chase again!
Path Compression

\[
\text{find}(x)
\]
\[
\quad \text{if } \pi(x) == x
\]
\[
\quad \quad \text{return } x
\]
\[
\quad \text{else}
\]
\[
\quad \quad \text{find}(\pi(x))
\]

What happens if we find\((x)\) again?
Chase again!

\[
\text{find}(x)
\]
\[
\quad \text{if } \pi(x) == x
\]
\[
\quad \quad \text{return } x
\]
\[
\quad \text{else}
\]
\[
\quad \quad \pi(x) = \text{find}(\pi(x))
\]
\[
\quad \quad \text{return } \pi(x)
\]
Is this better..

..asymptotically?
Is this better..

..asymptotically?

Take a deep breath.
Is this better.. 
..asymptotically?
Take a deep breath.
Fancy stuff..next!
Is this better...

..asymptotically?
Take a deep breath.
Fancy stuff..next!
Don’t worry.
Is this better..

..asymptotically?
Take a deep breath.
Fancy stuff..next!
Don’t worry.
...do try..
Is this better..

..asymptotically?
Take a deep breath.
Fancy stuff..next!
Don’t worry.
...do try..you’ll get smarter!
Path Compression Analysis

Union is same.
Path Compression Analysis

Union is same. Only affects root nodes.
Path Compression Analysis

Union is same. Only affects root nodes.
Rank properties still hold.

union(a,c) union(b,c) union(c,d)

find(a) \( \Theta(\log n) \) time for this find.

Do you see how?

(A) Yes
(B) No
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  rank to parent is higher
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  rank to parent is higher and \( \geq 2^k \) node in rank \( k \)
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  rank to parent is higher and $\geq 2^k$ node in rank $k$

Every find is asymptotically faster?
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  rank to parent is higher and $\geq 2^k$ node in rank $k$

Every find is asymptotically faster?

No. Can make a find take $\Theta(\log n)$ time.
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  rank to parent is higher and $\geq 2^k$ node in rank $k$

Every find is asymptotically faster?

No. Can make a find take $\Theta(\log n)$ time.

Do you see how?

(A) Yes

(B) No
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  rank to parent is higher and $\geq 2^k$ node in rank $k$

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(A) Yes
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Every find is asymptotically faster?
No. Can make a find take $\Theta(\log n)$ time.
Do you see how?
  (A) Yes
  (B) No

![Diagram](https://example.com/diagram.png)

```
union(a,c) union(b,c)  
...  
union subtree roots to build tree  
find(a)  
$$\Theta(\log n)$$ time for this find.
```
Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
  - rank to parent is higher and $\geq 2^k$ node in rank $k$

Every find is asymptotically faster?

No. Can make a find take $\Theta(\log n)$ time.

Do you see how?

(A) Yes

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Path Compression Analysis

Union is same. Only affects root nodes.

Rank properties still hold.
   rank to parent is higher and \( \geq 2^k \) node in rank \( k \)

Every find is asymptotically faster?

No. Can make a find take \( \Theta(\log n) \) time.

Do you see how?

(A) Yes
(B) No

\[
\text{union}(a,c) \quad \text{union}(b,c) \\
\quad \cdots \text{union}(c,d) \\
\text{union subtree roots to build tree} \\
\text{find}(a)
\]
Path Compression Analysis

Union is same. Only affects root nodes.
Rank properties still hold.
  rank to parent is higher and $\geq 2^k$ node in rank $k$
Every find is asymptotically faster?
No. Can make a find take $\Theta(\log n)$ time.
Do you see how?
(A) Yes
(B) No

union(a,c) union(b,c)
  … union(c,d)
union subtree roots to build tree
find(a)
$\Theta(\log n)$ time for this find.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.
$O(\log^* n)$ time on average!
Amortized Analysis.

Show that $m$ finds take $O(m\log^* n)$ time.

$O(\log^* n)$ time on average!

$log^* n$ is number of times one takes $\log$ to get to 1.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

(A) 4

(B) 2

(C) 3
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

(A) 4

(B) 2

(C) 3

C.
Amortized Analysis.

Show that \( m \) finds take \( O(m \log^* n) \) time.

\( O(\log^* n) \) time on average!

\( \log^* n \) is number of times one takes \( \log \) to get to 1.

\( \log^*(16) \)?

(A) 4

(B) 2

(C) 3

C. \( \log 16 = 4 \),
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$log^* n$ is number of times one takes $\log$ to get to 1.

$log^*(16)?$

(A) 4

(B) 2

(C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time. 

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

- (A) 4
- (B) 2
- (C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$log^* n$ is number of times one takes log to get to 1.

$log^*(16)$?

(A) 4

(B) 2

(C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.

Also $2^{2^2} = 16$. 
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

(A) 4

(B) 2

(C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.

Also $2^{2^2} = 16$. height of powers of two!
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

(A) 4

(B) 2

(C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.

Also $2^{2^2} = 16$. height of powers of two!

$\log 1,000,000$ versus $\log^* 1,000,000$?
Amortized Analysis.

Show that \( m \) finds take \( O(m \log^* n) \) time.

\( O(\log^* n) \) time on average!

\( \log^* n \) is number of times one takes \( \log \) to get to 1.

\( \log^*(16) \)?

(A) 4

(B) 2

(C) 3

C. \( \log 16 = 4, \log 4 = 2, \log 2 = 1 \). 3 times.

Also \( 2^{2^2} = 16 \). height of powers of two!

\( \log 1,000,000 \) versus \( \log^* 1,000,000 \)?

20 versus 5.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$log^* n$ is number of times one takes $\log$ to get to 1.

$log^*(16)$?

(A) 4

(B) 2

(C) 3

\[ \text{C. } \log 16 = 4, \log 4 = 2, \log 2 = 1. \] 3 times.

Also $2^{2^2} = 16$. height of powers of two!

$log 1,000,000$ versus $log^* 1,000,000$?

20 versus 5.

$log 1,000,000^{1,000,000}$ versus $log^* 1,000,000^{1,000,000}$?
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

(A) 4
(B) 2
(C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.

Also $2^2 = 16$. height of powers of two!

$\log 1,000,000$ versus $\log^* 1,000,000$?

20 versus 5.

$\log 1,000,000^{1,000,000}$ versus $\log^* 1,000,000^{1,000,000}$?

20,000,000 versus 6.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time.

$O(\log^* n)$ time on average!

$\log^* n$ is number of times one takes $\log$ to get to 1.

$\log^*(16)$?

(A) 4

(B) 2

(C) 3

C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.

Also $2^{2^2} = 16$. height of powers of two!

$\log 1,000,000$ versus $\log^* 1,000,000$?

20 versus 5.

$\log 1,000,000^{1,000,000}$ versus $\log^* 1,000,000^{1,000,000}$?

20,000,000 versus 6.

Grows very slowly.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time in total.
Amortized Analysis.

Show that \( m \) finds take \( O(m \log^* n) \) time in total.

\( O(\log^* n) \) time on average!
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time in total. $O(\log^* n)$ time on average!
Amortize cost = average over many operations.
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time in total.

$O(\log^* n)$ time on average!

Amortize cost = average over many operations.

Who else amortizes?
Show that $m$ finds take $O(m \log^* n)$ time in total.
$O(\log^* n)$ time on average!
Amortize cost = average over many operations.
Who else amortizes?
Bankers!
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time in total.

$O(\log^* n)$ time on average!

Amortize cost = average over many operations.

Who else amortizes?

Bankers!

Hand out some money
Amortized Analysis.

Show that $m$ finds take $O(m \log^* n)$ time in total.

$O(\log^* n)$ time on average!

Amortize cost = average over many operations.

Who else amortizes?

Bankers!

Hand out some money

..... use it to pay for each pointer change.
Amortized Analysis.

Show that \( m \) finds take \( O(m \log^* n) \) time in total. \( O(\log^* n) \) time on average!

Amortize cost = average over many operations.

Who else amortizes?

Bankers!

Hand out some money

..... use it to pay for each pointer change.

Only hand out \( O(m \log^* n) \) dollars.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
rank will no longer change!
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
rank will no longer change!

Divide non-zero ranks into levels.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
   rank will no longer change!

Divide non-zero ranks into levels.
\{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \cdots \{k + 1, \ldots 2^k\} \cdots
Handing out dollars.

Will hand out money to internal nodes to pay for them changing pointers in find.

Notice: When a node becomes an internal node, rank will no longer change!

Divide non-zero ranks into levels.

\{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \ldots \{k + 1, \ldots 2^k\} \ldots

How many groups of ranks?

(A) \(\Theta(\log n)\)

(B) \(\Theta(\log^* n)\)
Handing out dollars.

Will hand out money to internal nodes.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node. rank will no longer change!

Divide non-zero ranks into levels.

\[ \{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \ldots \{k + 1, \ldots 2^k\} \ldots \]

How many groups of ranks?

(A) \( \Theta(\log n) \)

(B) \( \Theta(\log^* n) \)

B.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
    rank will no longer change!

Divide non-zero ranks into levels.
\{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \cdots \{k + 1, \ldots 2^k\} \cdots

How many groups of ranks?

(A) \Theta(\log n)

(B) \Theta(\log^* n)

B. Each group grows by powering two!
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
rank will no longer change!

Divide non-zero ranks into levels.

\{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \cdots \{k+1, \ldots 2^k\} \cdots

How many groups of ranks?

(A) \(\Theta(\log n)\)

(B) \(\Theta(\log^* n)\)

B. Each group grows by powering two!

How many internal nodes ever get rank \(r\)?

(A) \(O(n/2^r)\)

(B) \(\Theta(n)\)
Handing out dollars.

Will hand out money to internal nodes .....to pay for them changing pointers in find.

Notice: When a node becomes an internal node. rank will no longer change!

Divide non-zero ranks into levels.
\{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \cdots \{k + 1, \ldots 2^k\} \cdots

How many groups of ranks?
(A) \(\Theta(\log n)\)
(B) \(\Theta(\log^* n)\)

B. Each group grows by powering two!

How many internal nodes ever get rank \(r\)?
(A) \(O(n/2^r)\)
(B) \(\Theta(n)\)

A.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node. rank will no longer change!

Divide non-zero ranks into levels.
\{1\}, \{2, 3, 4\}, \{5, \ldots, 16\} \ldots \{k + 1, \ldots 2^k\} \ldots

How many groups of ranks?

(A) \(\Theta(\log n)\)
(B) \(\Theta(\log^* n)\)

B. Each group grows by powering two!

How many internal nodes ever get rank \(r\)?

(A) \(O(n/2^r)\)
(B) \(\Theta(n)\)

A. Each contained \(\geq 2^r\) nodes when root.
Handing out dollars.

Will hand out money to internal nodes
.....to pay for them changing pointers in find.

Notice: When a node becomes an internal node.
   rank will no longer change!

Divide non-zero ranks into levels.
{1}, {2,3,4}, {5, ..., 16} · · · {k + 1, ..., 2^k} · · ·

How many groups of ranks?
(A) Θ(log n)
(B) Θ(log^* n)

B. Each group grows by powering two!

How many internal nodes ever get rank r?
(A) O(n/2^r)
(B) Θ(n)

A. Each contained ≥ 2^r nodes when root. Separate nodes.
Handing out money!

Will hand out money to internal nodes

Notice: When a node becomes an internal node, its rank will no longer change!

If in a set of ranks \(\{k+1, \ldots, 2k\}\), give node 2\(k\) dollars.

\[O\left(\frac{n}{2^k}\right)\] internal nodes of rank \(r\).

Total Doled out:

In a group:

\[2^k \left(\frac{n}{2^k} + 1 + \frac{n}{2^k} + 2 + \cdots + \frac{n}{2^{2k}}\right) = O\left(\frac{n}{2}\right)\]

\[O\left(\log^* n\right)\] groups. Total money:

\[O\left(n \log^* n\right)\].
Handing out money!

Will hand out money to internal nodes
.....since they change pointers in find.
Handing out money!

Will hand out money to internal nodes
.....since they change pointers in find.

Notice: When a node becomes an internal node.
rank will no longer change!
Handing out money!

Will hand out money to internal nodes
.....since they change pointers in find.

Notice: When a node becomes an internal node.
  rank will no longer change!

If in set of ranks \( \{k + 1, \ldots, 2^k\} \) give node \( 2^k \) dollars.
Handing out money!

Will hand out money to internal nodes
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Notice: When a node becomes an internal node.
    rank will no longer change!

If in set of ranks \( \{ k + 1, \ldots, 2^k \} \) give node \( 2^k \) dollars.

\( O(n/2^r) \) internal nodes of rank \( r \).
Handing out money!

Will hand out money to internal nodes
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Notice: When a node becomes an internal node.
    rank will no longer change!
If in set of ranks \( \{k + 1, \ldots, 2^k\} \) give node \( 2^k \) dollars.

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Total Doled out:
Handing out money!

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Total Doled out:
    In a group:
Handing out money!

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If in set of ranks \( \{k + 1, \ldots, 2^k\} \) give node \( 2^k \) dollars.

\( O(n/2^r) \) internal nodes of rank \( r \).

Total Doled out:
    In a group: \( 2^k (n/2^{k+1} + n/2^{k+2} \cdots + n/2^{2k}) = O(n) \).
Handing out money!

Will hand out money to internal nodes
.....since they change pointers in find.

Notice: When a node becomes an internal node.
rank will no longer change!

If in set of ranks \( \{k + 1, \ldots, 2^k\} \) give node \( 2^k \) dollars.

\( O(n/2^r) \) internal nodes of rank \( r \).

Total Doled out:
In a group: \( 2^k \left( n/2^{k+1} + n/2^{k+2} \cdots + n/2^{2k} \right) = O(n) \).
\( O(\log^* n) \) groups. **Total money: \( O(n \log^* n) \).**
Bounding find cost.

Bound cost of find operation.
Bounding find cost.

Bound cost of find operation.

$rank(x)$ and $rank(r)$ in different groups.
Bounding find cost.

Bound cost of find operation.

$\text{rank}(x)$ and $\text{rank}(r)$ in different groups.

$\text{rank}(z)$ and $\text{rank}(r)$ in same group.
Bounding find cost.

Bound cost of find operation.

\[ x \rightarrow z \rightarrow r \]

\( \text{rank}(x) \) and \( \text{rank}(r) \) in different groups.

\( \text{rank}(z) \) and \( \text{rank}(r) \) in same group.
Bounding find cost.

Bound cost of find operation.

\[ O(1) \text{ plus } \]

\[ \text{ rank}(x) \text{ and } \text{ rank}(r) \text{ in different groups.} \]

\[ \text{ rank}(z) \text{ and } \text{ rank}(r) \text{ in same group.} \]
Bounding find cost.

Bound cost of find operation.

$\text{rank}(x)$ and $\text{rank}(r)$ in different groups.

$\text{rank}(z)$ and $\text{rank}(r)$ in same group.

$O(1)$ plus

cost of changing pointers to point to higher ranked nodes
Bounding find cost.

Bound cost of find operation.

\[ O(1) \] plus cost of changing pointers to point to higher ranked nodes

Per Operation part.

\[ \text{rank}(x) \] and \( \text{rank}(r) \) in different groups.

\[ \text{rank}(z) \] and \( \text{rank}(r) \) in same group.
Bounding find cost.

Bound cost of find operation.

\[ x \xrightarrow{z} r \]

\( \text{rank}(x) \) and \( \text{rank}(r) \) in different groups.
\( \text{rank}(z) \) and \( \text{rank}(r) \) in same group.

\( O(1) \) plus

\[ \text{cost of changing pointers to point to higher ranked nodes} \]

Per Operation part.

\( O(\log^* n) \) pointers that point to node to a higher group.
Bounding find cost.

Bound cost of find operation.

\[ \text{rank}(x) \text{ and } \text{rank}(r) \text{ in different groups.} \]
\[ \text{rank}(z) \text{ and } \text{rank}(r) \text{ in same group.} \]

\[ O(1) \] plus

\[ \text{cost of changing pointers to point to higher ranked nodes} \]

Per Operation part.

\[ O(\log^* n) \] pointers that point to node to a higher group.

Total per operation cost over \( m \) finds: \( O(m \log^* n) \).
Bounding find cost.

Bound cost of find operation.

\[ \text{rank}(x) \text{ and } \text{rank}(r) \text{ in different groups.} \]

\[ \text{rank}(z) \text{ and } \text{rank}(r) \text{ in same group.} \]

\[ \text{Per Operation part.} \]

\[ O(1) \text{ plus } O(\log^* n) \text{ pointers that point to node to a higher group.} \]

Total per operation cost over \( m \) finds: \( O(m \log^* n) \).
Bounding find cost.

Bound cost of find operation.

\[ x \xrightarrow{\cdot} z \xrightarrow{\cdot} r \]

- \(\text{rank}(x)\) and \(\text{rank}(r)\) in different groups.
- \(\text{rank}(z)\) and \(\text{rank}(r)\) in same group.

\(O(1)\) plus

- cost of changing pointers to point to higher ranked nodes

Per Operation part.

- \(O(\log^* n)\) pointers that point to node to a higher group.

Total per operation cost over \(m\) finds: \(O(m\log^* n)\).

Amortized Part.
Bounding find cost.

Bound cost of find operation.

$\text{rank}(x)$ and $\text{rank}(r)$ in different groups.

$\text{rank}(z)$ and $\text{rank}(r)$ in same group.

$O(1)$ plus cost of changing pointers to point to higher ranked nodes

**Per Operation part.**

$O(\log^* n)$ pointers that point to node to a higher group.

Total per operation cost over $m$ finds: $O(m \log^* n)$.

**Amortized Part.**

Node uses its dollars to pay for changing a pointer within group.
Bounding find cost.

Bound cost of find operation.

\[ x \rightarrow z \rightarrow r \]

*rank*(\(x\)) and *rank*(\(r\)) in different groups.

*rank*(\(z\)) and *rank*(\(r\)) in same group.

\(O(1)\) plus

- cost of changing pointers to point to higher ranked nodes

**Per Operation part.**

\(O(\log^* n)\) pointers that point to node to a higher group.

Total per operation cost over \(m\) finds: \(O(m \log^* n)\).

**Amortized Part.**

Node uses its dollars to pay for changing a pointer within group.

Recall group: \(\{k+1, \ldots, 2^{k+1}\}\)
Bounding find cost.
Bound cost of find operation.

\[
\begin{align*}
\text{rank}(x) & \quad \text{and} \quad \text{rank}(r) \quad \text{in different groups.} \\
\text{rank}(z) & \quad \text{and} \quad \text{rank}(r) \quad \text{in same group.}
\end{align*}
\]

\(O(1)\) plus

- cost of changing pointers to point to higher ranked nodes

Per Operation part.
\(O(\log^* n)\) pointers that point to node to a higher group.
Total per operation cost over \(m\) finds: \(O(m \log^* n)\).

Amortized Part.
Node uses its dollars to pay for changing a pointer within group.

Recall group: \(\{k + 1, \ldots, 2^{k+1}\}\)

Enough money?
Bounding find cost.

Bound cost of find operation.

$rank(x)$ and $rank(r)$ in different groups.

$rank(z)$ and $rank(r)$ in same group.

$O(1)$ plus

\[
O(\log^* n) \text{ pointers that point to node to a higher group.}
\]

Total per operation cost over $m$ finds: $O(m\log^* n)$.

Amortized Part.

Node uses its dollars to pay for changing a pointer within group.

Recall group: $\{k+1, \ldots, 2^{k+1}\}$

- Enough money?
  - only $2^k$ ranks in group.
Bounding find cost.

Bound cost of find operation.

- $rank(x)$ and $rank(r)$ in different groups.
- $rank(z)$ and $rank(r)$ in same group.

$O(1)$ plus

- cost of changing pointers to point to higher ranked nodes

Per Operation part.

- $O(\log^* n)$ pointers that point to node to a higher group.

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Recall group: $\{k + 1, \ldots, 2^{k+1}\}$

- Enough money?
  - only $2^k$ ranks in group.
  - each node in group has $2^k$ dollars.
Bounding find cost.

Bound cost of find operation.

\[ x \rightarrow z \rightarrow r \]

*rank*(\(x\)) and *rank*(\(r\)) in different groups.

*rank*(\(z\)) and *rank*(\(r\)) in same group.

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**Per Operation part.**

\[ O(\log^* n) \] pointers that point to node to a higher group.

Total per operation cost over \(m\) finds: \(O(m \log^* n)\).

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Recall group: \(\{k + 1, \ldots, 2^{k+1}\}\)

Enough money?
- only \(2^k\) ranks in group.
- each node in group has \(2^k\) dollars.  Enough money!
Bounding find cost.

Bound cost of find operation.

\[ rank(x) \text{ and } rank(r) \text{ in different groups.} \]

\[ rank(z) \text{ and } rank(r) \text{ in same group.} \]

\( O(1) \) plus

cost of changing pointers to point to higher ranked nodes

Per Operation part.

\( O(\log^* n) \) pointers that point to node to a higher group.

Total per operation cost over \( m \) finds: \( O(m \log^* n) \).

Amortized Part.

Node uses its dollars to pay for changing a pointer within group.

Recall group: \( \{k + 1, \ldots, 2^{k+1}\} \)

Enough money?

only \( 2^k \) ranks in group.

each node in group has \( 2^k \) dollars. \hspace{1cm} \text{Enough money!} \)

Total money: \( O(n \log^* n) \).
Bounding find cost.

Bound cost of find operation.

\[ x \rightarrow z \rightarrow r \]

\( rank(x) \) and \( rank(r) \) in different groups.
\( rank(z) \) and \( rank(r) \) in same group.

\( O(1) \) plus

cost of changing pointers to point to higher ranked nodes

Per Operation part.
\( O(\log^* n) \) pointers that point to node to a higher group.
Total per operation cost over \( m \) finds: \( O(m \log^* n) \).

Amortized Part.
Node uses its dollars to pay for changing a pointer within group.

Recall group: \( \{k + 1, \ldots, 2^{k+1} \} \)

Enough money?
only \( 2^k \) ranks in group.
each node in group has \( 2^k \) dollars. Enough money!

Total money: \( O(n \log^* n) \).

\[ \Rightarrow \] Total find cost:
Bounding find cost.

Bound cost of find operation.

\[ x \rightarrow z \rightarrow r \]

\( rank(x) \) and \( rank(r) \) in different groups.

\( rank(z) \) and \( rank(r) \) in same group.

\( O(1) \) plus

\[ \text{cost of changing pointers to point to higher ranked nodes} \]

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\( O(\log^* n) \) pointers that point to node to a higher group.

Total per operation cost over \( m \) finds: \( O(m \log^* n) \).

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Node uses its dollars to pay for changing a pointer within group.

Recall group: \( \{k + 1, \ldots, 2^{k+1}\} \)

Enough money?

- only \( 2^k \) ranks in group.
- each node in group has \( 2^k \) dollars.

Enough money!

Total money: \( O(n \log^* n) \). \( \implies \) Total find cost: \( O((m + n) \log^* n!) \)
Bounding find cost.

Bound cost of find operation.

\[
\text{rank}(x) \text{ and rank}(r) \text{ in different groups.}
\]
\[
\text{rank}(z) \text{ and rank}(r) \text{ in same group.}
\]

\[O(1)\] plus

cost of changing pointers to point to higher ranked nodes

**Per Operation part.**
\[O(\log^* n)\] pointers that point to node to a higher group.

Total per operation cost over \(m\) finds: \(O(m \log^* n)\).

**Amortized Part.**

Node uses its dollars to pay for changing a pointer within group.

Recall group: \(\{k + 1, \ldots, 2^{k+1}\}\)

Enough money?

only \(2^k\) ranks in group.

each node in group has \(2^k\) dollars.  

Enough money!

Total money: \(O(n \log^* n)\).  \(\implies\) Total find cost: \(O((m + n) \log^* n)\)!
Bounding find cost.

Bound cost of find operation.

\[ x \rightarrow z \rightarrow r \]

*rank*(*x*) and *rank*(*r*) in different groups.

*rank*(*z*) and *rank*(*r*) in same group.

\[ O(1) \] plus

- cost of changing pointers to point to higher ranked nodes

Per Operation part.

\[ O(\log^* n) \] pointers that point to node to a higher group.

Total per operation cost over *m* finds: \( O(m \log^* n) \).

Amortized Part.

Node uses its dollars to pay for changing a pointer within group.

Recall group: \( \{k + 1, \ldots, 2^{k+1}\} \)

Enough money?

- only \( 2^k \) ranks in group.
- each node in group has \( 2^k \) dollars.  
  Enough money!

Total money: \( O(n \log^* n) \).  \( \implies \) Total find cost: \( O((m + n) \log^* n)! \)
Bounding find cost.

Bound cost of find operation.

\[ \text{rank}(x) \text{ and } \text{rank}(r) \text{ in different groups.} \]

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Recall group: \( \{ k + 1, \ldots, 2^{k+1} \} \)

Enough money?

only \( 2^k \) ranks in group.

each node in group has \( 2^k \) dollars.  Enough money!

Total money: \( O(n \log^* n) \).  \( \implies \) Total find cost: \( O((m + n) \log^* n)! \) ! ! !
Bounding find cost.

Bound cost of find operation.

\[ \text{rank}(x) \text{ and } \text{rank}(r) \text{ in different groups.} \]

\[ \text{rank}(z) \text{ and } \text{rank}(r) \text{ in same group.} \]

\[ O(1) \text{ plus } \]

cost of changing pointers to point to higher ranked nodes

Per Operation part.

\[ O(\log^* n) \] pointers that point to node to a higher group.

Total per operation cost over \( m \) finds: \( O(m \log^* n) \).

Amortized Part.

Node uses its dollars to pay for changing a pointer within group.

Recall group: \( \{k + 1, \ldots, 2^{k+1}\} \)

Enough money?

only \( 2^k \) ranks in group.

each node in group has \( 2^k \) dollars. Enough money!

Total money: \( O(n \log^* n) \). \( \implies \) Total find cost: \( O((m + n) \log^* n)! \)
Bounding find cost.

Bound cost of find operation.

\[ \text{rank}(x) \text{ and } \text{rank}(r) \text{ in different groups.} \]

\[ \text{rank}(z) \text{ and } \text{rank}(r) \text{ in same group.} \]

\[ O(1) \text{ plus } \]

\[ \text{cost of changing pointers to point to higher ranked nodes} \]

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- only \( 2^k \) ranks in group.
- each node in group has \( 2^k \) dollars. \( \Rightarrow \) Enough money!

Total money: \( O(n \log^* n) \). \( \Rightarrow \) Total find cost: \( O((m + n) \log^* n) ! ! ! ! ! \)
Bounding find cost.

Bound cost of find operation.

$\text{rank}(x)$ and $\text{rank}(r)$ in different groups.

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$O(1)$ plus cost of changing pointers to point to higher ranked nodes

Per Operation part.

$O(\log^* n)$ pointers that point to node to a higher group.

Total per operation cost over $m$ finds: $O(m \log^* n)$.

Amortized Part.

Node uses its dollars to pay for changing a pointer within group.

Recall group: $\{k + 1, \ldots, 2^{k+1}\}$

Enough money?

- only $2^k$ ranks in group.
- each node in group has $2^k$ dollars. Enough money!

Total money: $O(n \log^* n)$. $\implies$ Total find cost: $O((m + n) \log^* n)$! ! ! ! !
Instant Replay

Intuition:
Instant Replay

Intuition:
Some operations may be expensive.
Intuition:

Some operations may be expensive.
...but modify data structure so they won’t be in future.
Instant Replay

Intuition:

Some operations may be expensive.
...but modify data structure so they won’t be in future.
    (Path Compression!)
Instant Replay

Intuition:

Some operations may be expensive.
...but modify data structure so they won’t be in future.
   (Path Compression!)

Place credits in data structure to pay for these modifications.
Intuition:

Some operations may be expensive.
...but modify data structure so they won’t be in future.
   (Path Compression!)

Place credits in data structure to pay for these modifications.

Still..
Instant Replay

Intuition:

Some operations may be expensive.
...but modify data structure so they won’t be in future.
   (Path Compression!)

Place credits in data structure to pay for these modifications.

Still..
   fancy business.
Ackerman’s function.

Can we do better?
Ackerman’s function.

Can we do better?

\[
f(k) \text{ is } 2^{2^k} \text{ of height } k.
\]
Ackerman’s function.

Can we do better?

$f(k)$ is $2^{2^k}$ of height $k$.

Grows fast.
Ackerman’s function.

Can we do better?

\[ f(k) \text{ is } 2^{2^k} \text{ of height } k. \]

Grows fast.

\[ f^{-1}(n) \text{ grows slowly!} \]
Ackerman’s function.

Can we do better?

::

\( f(k) \) is \( 2^{2^k} \) of height \( k \).

Grows fast.

\( f^{-1}(n) \) grows slowly! For example \( f^{-1}((10^6)^{10^6}) = 5 \).
Can we do better?

\[ f(k) \text{ is } 2^{2^k} \text{ of height } k. \]

Grows fast.

\[ f^{-1}(n) \text{ grows slowly! For example } f^{-1}((10^6)^{10^6}) = 5. \]

Ackermans function grows even faster: computable but grows faster than any primitive recursive function.
Ackerman’s function.

Can we do better?

\[ f(k) \text{ is } 2^{2^k} \text{ of height } k. \]

Grows fast.

\[ f^{-1}(n) \text{ grows slowly! For example } f^{-1}((10^6)^{10^6}) = 5. \]

Ackermans function grows even faster: computable but grows faster than any primitive recursive function.

There is MST algorithm that runs in \( O(m\alpha(m, n)) \) where \( \alpha(m, n) \) is inverse Ackerman’s function.
Lecture in a minute.

Horn Formula:
- Implications of positive literals with ANDs on one side.
- Plus ORs of negatives.
- Negative clauses problem only with true literals.
- Greedy: only set true if have to.
Lecture in a minute.

Horn Formula:
- Implications of positive literals with ANDs on one side.
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- Greedy: only set true if have to.

Set Cover:
- Given subsets of some elements.
- Find: min number of sets that contains every element.
- Greedy: choose largest set w.r.t. remaining elements.
- $O(\log n)$ approximate solution.
- Proof Idea: optimal of size $k \Rightarrow$ Cover $1/k$ of the remaining elts.

Path Compression:
- $O(m \log^* n)$ time for $m$ finds.
- Some finds expensive but cheap on average.
- Idea: group ranks into $\log^* n$ sets.
- Small number of pointers across sets in any find.
- Total movement inside sets $O(n)$.
- Idea: from not more than 2$^k$ nodes of rank $k$. 
Lecture in a minute.

Horn Formula:
- Implications of positive literals with ANDs on one side.
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