CS 170
Efficient Algorithms and Intractable Problems

Lecture 13
Dynamic Programming III

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Announcements

Interested in meeting 1-1 with TAs?
→ Fill out a form on Ed
→ General advice for course, midterm performance, and etc.

Recap of the last 2 lectures

Dynamic Programming!

The recipe!

Step 1. Identify subproblems (aka optimal substructure)
Step 2. Find a recursive formulation for the subproblems
Step 3. Design the Dynamic Programming Algorithm
→ Memo-ize computation starting from smallest subproblems and building up.

We saw a lot of examples already
→ Fibonacci
→ Shortest Paths (in DAGs, Bellman-Ford, and All-Pair)
→ Longest increasing subsequence
→ Edit distance

This lecture

Even more examples!

→ Knapsack (without repetition)

→ Traveling Salesman Problem

→ Independent Sets on Trees

Best way to learn dynamic programming is by doing a lot of examples!

By doing more examples today, we will also develop intuition about how to choose subproblems (Recipe's step 1).

Knapsack

Knapsack

All integers!

Input: A weight capacity W, and n items with (weights, values), $(w_1, v_1), \cdots, (w_n, v_n)$.

Output: Most valuable combination of items, whose total weight is at most W.

Two variants:

1. With repetition (aka unbounded supply, aka with replacement)

→ For each item i, we can take as many copies of it as we want

2. Without repetition (0-1 knapsack, aka without replacement)

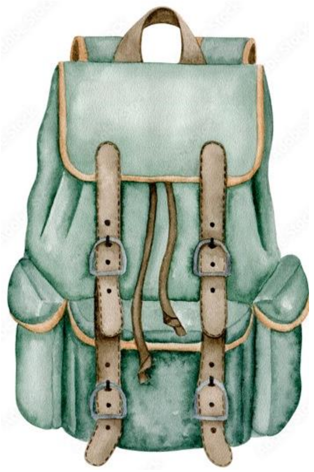
→ For each item, either we take 1 copy or 0 copy of it.

Knapsack

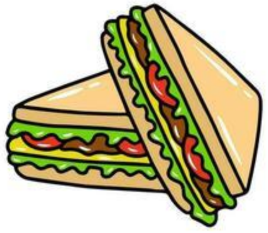
All integers!

Input: A weight capacity W, and n items with (weights, values), $(w_1, v_1), \cdots, (w_n, v_n)$.

Output: Most valuable combination of items, whose total weight is at most W.



W = 10

Item				
Weight:	6	3	4	2
Value:	30	14	16	9

With repetition:
1 tent + 2 sandwiches = **48 value**
Weight =10

Without repetition:
1 tent + 1 stove = **46 value**
Weight =10

Step 1: Subproblems of Knapsack (with repetition)

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

Output: Most valuable combination of items (with repetition), whose total weight is ≤W.

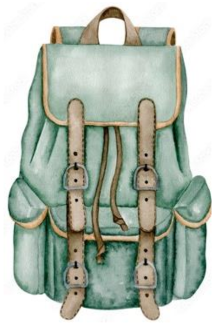
What makes for good subproblems?

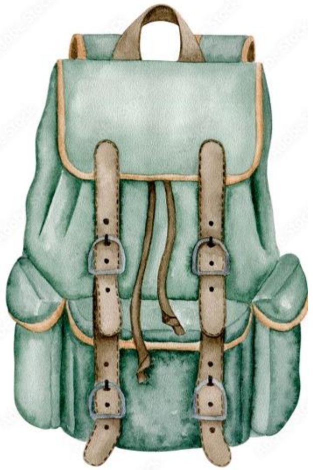
- Not too many of them (the more subproblems the slower the DP algorithm)
- Must have enough information in it to compute subproblems recursively (needed for step 2).

Subproblems: For all $c \leq W$, $K(c) =$ best value achievable for knapsack of capacity c.

First solve the problem for small knapsacks

Then larger knapsacks





Step 2: Recurrence in Knapsack (with repetition)

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

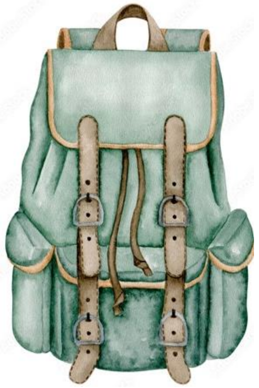
Output: Most valuable combination of items (with repetition), whose total weight is ≤W.

Step 1: Subproblems $K(c)$ = best value achievable for knapsack of capacity c, for c $\leq W$.
Step 2:
Let's say we commit to putting a copy of item i for which $w_i \leq c$ in the knapsack
→ Then only $c - w_i$ capacity remains to be optimally packed.
→ The recurrence relationship

value item i

$$K(c) = \max_{i: w_i \leq c} \{v_i + K(c - w_i)\}$$

optimal value of remaining capacity



Step 3: Design the Algorithm

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

Output: Most valuable combination of items (with repetition), whose total weight is ≤W.

How do we memo-ize the subproblems in this recurrence relation?

$$K(c) = \max_{i: w_i \le c} \{v_i + K(c - w_i)\}$$

$c \le w$

Runtime of this algorithm?

Number of subproblems: $O(W)$

Per subproblem, max over $O(n)$ cases
→ $O(n)$ time per subproblem.

Total runtime: $O(nW)$

Knapsack-with-repetition$(W, (w_1, v_1), \ldots, (w_n, v_n))$

An array K of size $W + 1$.

$K[0] = 0$

$O(n)$

For $c = 1, \ldots, W$,

$\boxed{K[c] = \max_{\boxed{i: w_i \le c}} \{v_i + K(c - w_i)\}}$

return $K[W]$

Polynomial vs Pseudo-Polynomial Time

We quantify runtimes as functions of input size.

→ **Input size**: # bits needed to write the input

$\underbrace{\text{Knapsack}\left(W, \{(v_i, w_i)\}_{i\in[n]}\right)}$

$\Downarrow$

$\log(W)$ $n\log(W)$

What is the input size the of Knapsack

- Weight capacity W → Needs $O(\log(W))$ bits

integer $1, \ldots, W$ ~~$W+1 \ldots$~~

- n items with weights at most W (remove any larger item) → most $O(\log(W))$ bits
- Total input size of knapsack: $O(n \log(W))$

Does the dynamic programming for knapsack run efficiently?

→ Not polynomial time exactly! **Runtime $O(nW)$** but **input size $O(n\, log(W))$**

numerical.

→ Called a pseudo-polynomial time algorithm

→ A runtime that's polynomial in the numerical value of the input (like W) but not in the size of the input (like $O(n\, log(W))$).

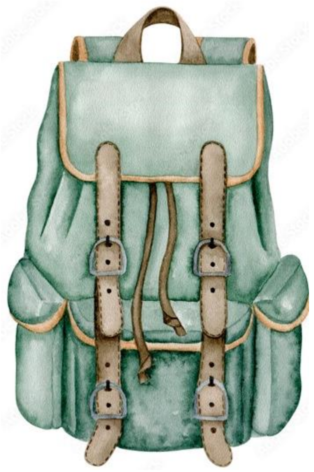
Knapsack without Repitions

Knapsack Recap

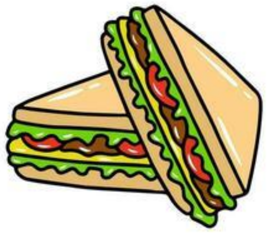
All integers!

Input: A weight capacity W, and n items with (weights, values), $(w_1, v_1), \cdots, (w_n, v_n)$.

Output: Most valuable combination of items, whose total weight is at most W.



W = 10

Item				
Weight:	6	3	4	2
Value:	30	14	16	9

Last Variant

With repetition:
1 tent + 2 sandwiches = **48 value**
Weight =10

This Variant

Without repetition:
1 tent + 1 stove = **46 value**
Weight =10

Step 1: Knapsack Subproblems

Can we still use the same subproblems

$K(c)$ = best value achievable for knapsack of capacity c, for c $\leq W$?

Challenge: We are only allowed one copy of an item, so the subproblem needs to "know" what items we have used and what we haven't.

We need a different way of tracking subproblems!

Idea: Solve knapsack for

- smaller sets of items and smaller capacities!

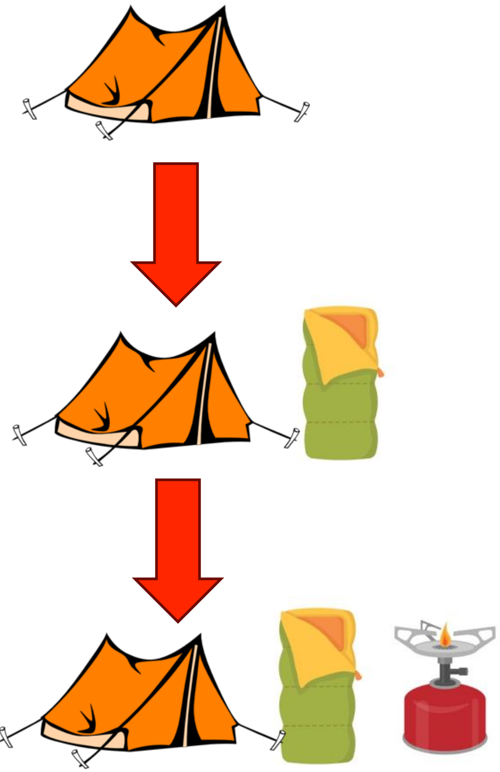
Step 1: Knapsack Subproblems (without repetition)

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

Output: Most valuable subset of items, whose total weight is $\leq$W.

First solve the problem for small knapsacks and small sets of items

Then larger knapsacks



And larger item sets

Step 2: Knapsack Recurrence (without repetition)

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

Output: Most valuable subset of items, whose total weight is $\leq$W.

Step1: Subproblems: For all $c \leq W$ and all $j \leq n$

$K(j, c)$ = best value achievable for knapsack of capacity c using only items $1, \ldots, j$

Discuss

Step 2: Compute $K(j, c)$ using smaller subproblems.

Case 1

Optimal solution using items $1, \ldots, j$ doesn't actually use item j.

Case 2

Optimal solution using items $1, \ldots, j$ uses item j.

$$K(j,c) = \max\{K(j-1,c), \quad K(j-1, c-w_j)+v_j\}$$

Hint: keep track of value, leftover capacity, and item set.

Step 3: Design the Algorithm

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

Output: Most valuable subset of items, whose total weight is $\leq$W.

How do we memo-ize the subproblems in this recurrence relation?

$$K(j, c) = \max_{j: w_j \leq c} \{ K(j-1, c), v_j + K(j-1, c-w_j) \}, \text{ base cases: } K(0, c) = 0 \text{ and } K(j, 0) = 0$$

	0	…	$c - w_j$	…	c	…	W
0							
⋮							
$j-1$			$K(j-1, c-w_j)$	…	$K(j-1, c)$		
j					$\boxed{K(j, c)}$		
⋮							
n							

Runtime of this algorithm

Input: A weight capacity W, and n items $(w_1, v_1), \cdots, (w_n, v_n)$. All integers.

Output: Most valuable subset of items, whose total weight is $\leq$W.

$O(nW)$ number of subproblems.

For each subproblem, we take max of 2 values:
→ Work per subproblem $O(1)$

Total runtime: $O(nW)$. pseudo-polynomial time alg

Space complexity: $O(nW)$

Knapsack-no-rep$(W, (w_1, v_1), \ldots, (w_n, v_n))$

An array K of size $(n+1) \times (W+1)$

For $c = 0, \ldots, W$: $K[0, c] = 0$

For $j = 0, \ldots, n$: $K[j, 0] = 0$

For $j = 1, \ldots, n$:

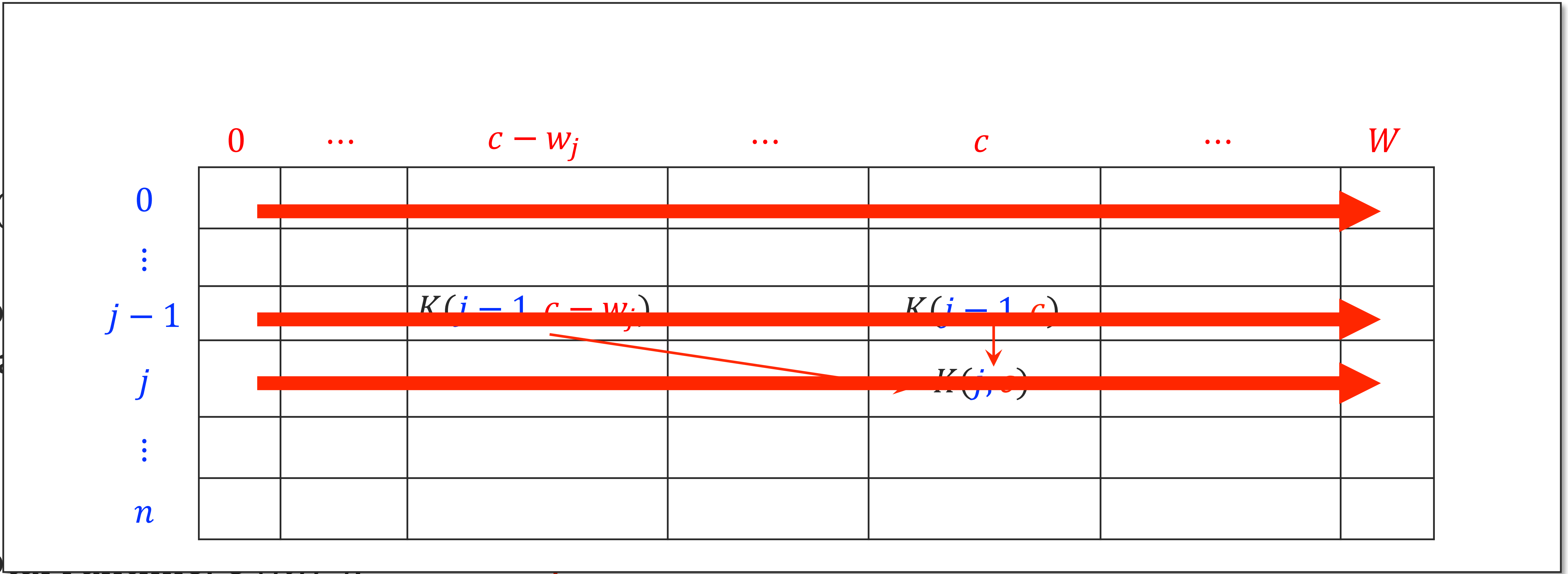
For $c = 1, \ldots, W$,

If j : $w_j \leq c$

$K[j, c] = \max_{\cancel{j}} \{ K(j-1, c), v_j + K(j-1, c - w_j) \}$

return $K[n, W]$

Runtime of this algorithm



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Space complexity: $\cancel{O(nW)}$ $O(W)$

$$K[j,c] = \max_{j: w_j < c} \{ K(j-1, c), v_j + K(j-1, c-w_j) \}$$

return $K[n, W]$

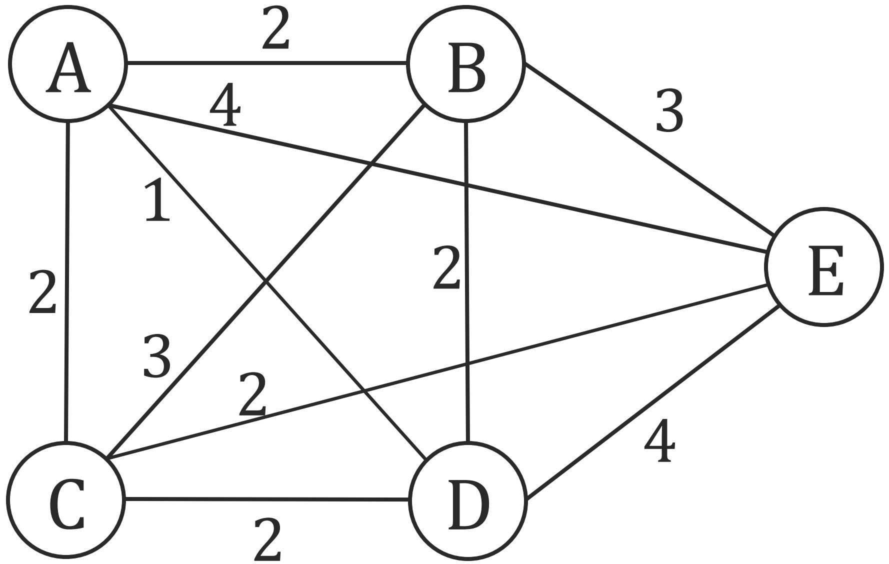
Traveling Salesperson Problem

Traveling Salesperson Problem (TSP)

Input: cities $1 \dots n$ and pairwise distances d_{ij} between cities i and j.

Output: A “tour” of minimum total distance.

Definition: A **tour** is a path through the cities, that
1) Starts from city 1
2) Visits every city, exactly once
3) Returns to city 1



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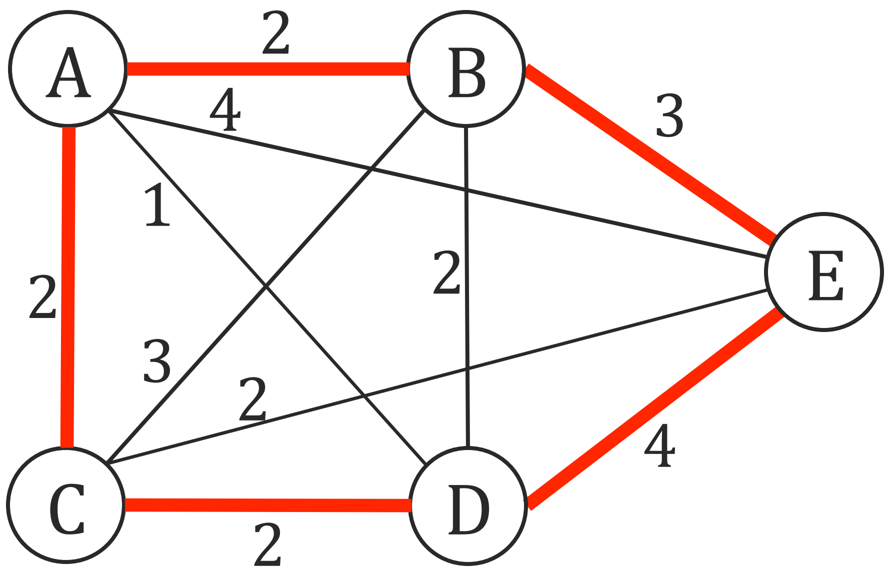
Tour of distance: 13



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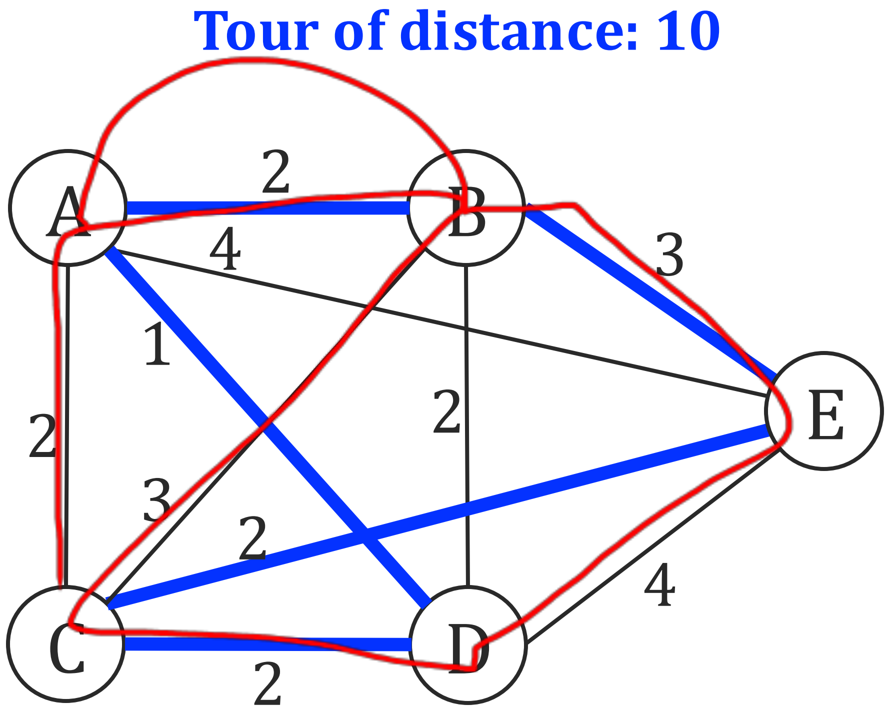

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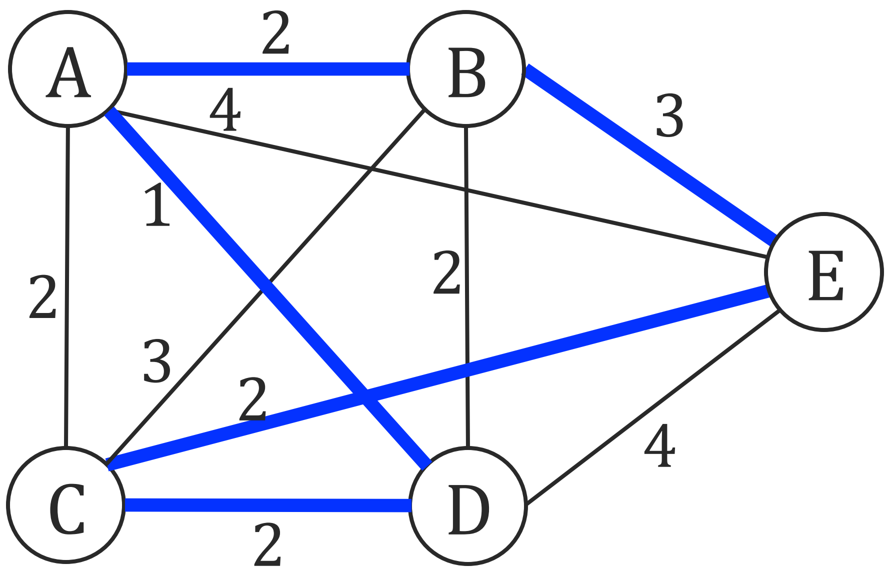
Tour of distance: 10



Naïve brute force algorithm:
→ $(n-1)!$ Tours
→ Each $O(n)$ to compute distance.
→ $O(n!)$ runtime $\approx n^n$

Dynamic programming gives us $O(n^2 2^n)$

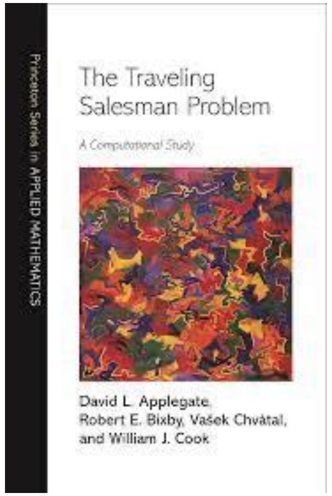
One of the most famous Math/CS problems.

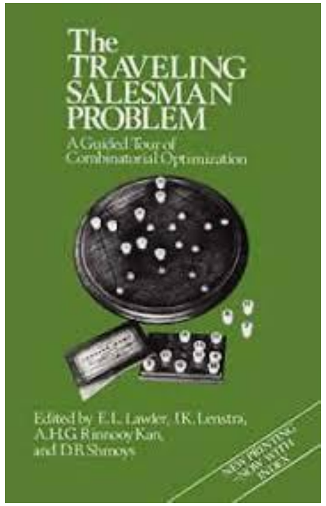
Notoriously difficult.

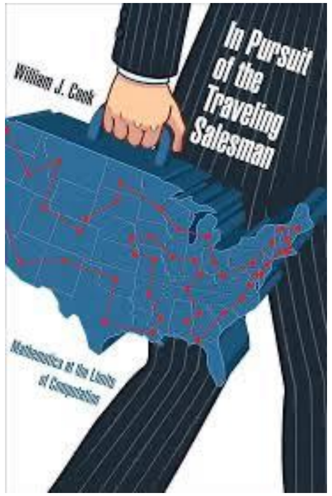
The DP algorithm is a substantial improvement over brute force. Take $n = 25$

$\rightarrow O(n!) \approx 10^{25}$

$\rightarrow O(n^2 2^n) \approx 10^{10}$







Step 1: Subproblems of TSP

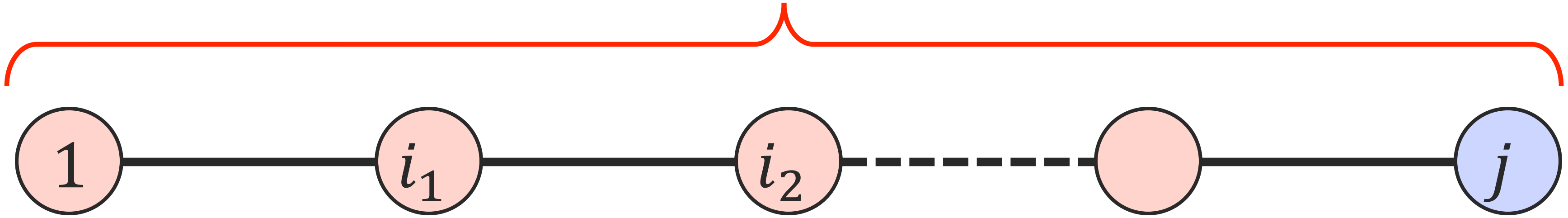
Input: cities $1 \ldots n$ and pairwise distances d_{ij} between cities i and j.

Output: A "tour" of minimum total distance.

Think of subproblems as partial tour!

→ It starts from city 1, ends in city j, and passing through all cities in a set S (which includes 1 and j).

Set S of cities (including 1 and j)



Subproblems: For all $j \leq n$ and $S \subseteq \{1, \ldots, n\}$, s.t. S includes 1 and j.

$T[S, j]$ = length of the shortest path visiting all cities in S exactly once, starting from 1 and ending at j.

Step 2: Recurrence Relation for TSP

Input: cities $1 \ldots n$ and pairwise distances d_{ij} between cities i and j.

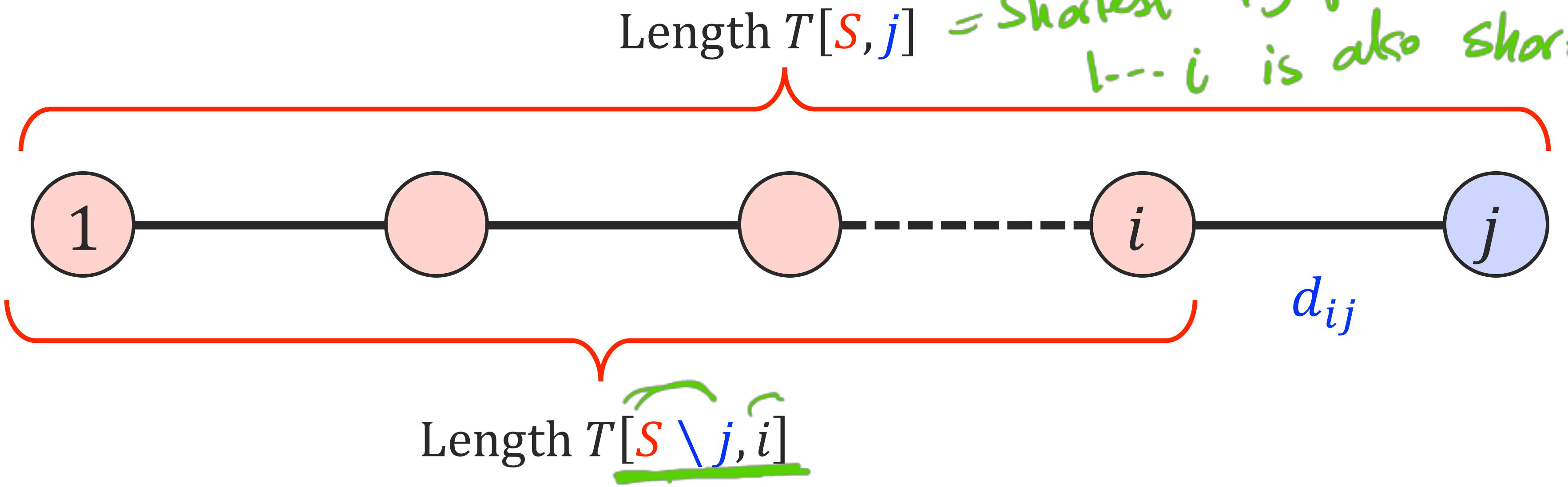
Output: A "tour" of minimum total distance.

Subproblems: For all $j \leq n$ and $S \subseteq \{1, \ldots, n\}$, s.t. S includes 1 and j.

$\boxed{T[S, j]}$ = length of the shortest path visiting all cities in S exactly once, starting from 1 and ending at j.

Step 2: Compute $T[S, j]$ using smaller subproblems.

= Shortest 1, j path.
1 --- i is also shortest.



Step 2: Recurrence Relation for TSP

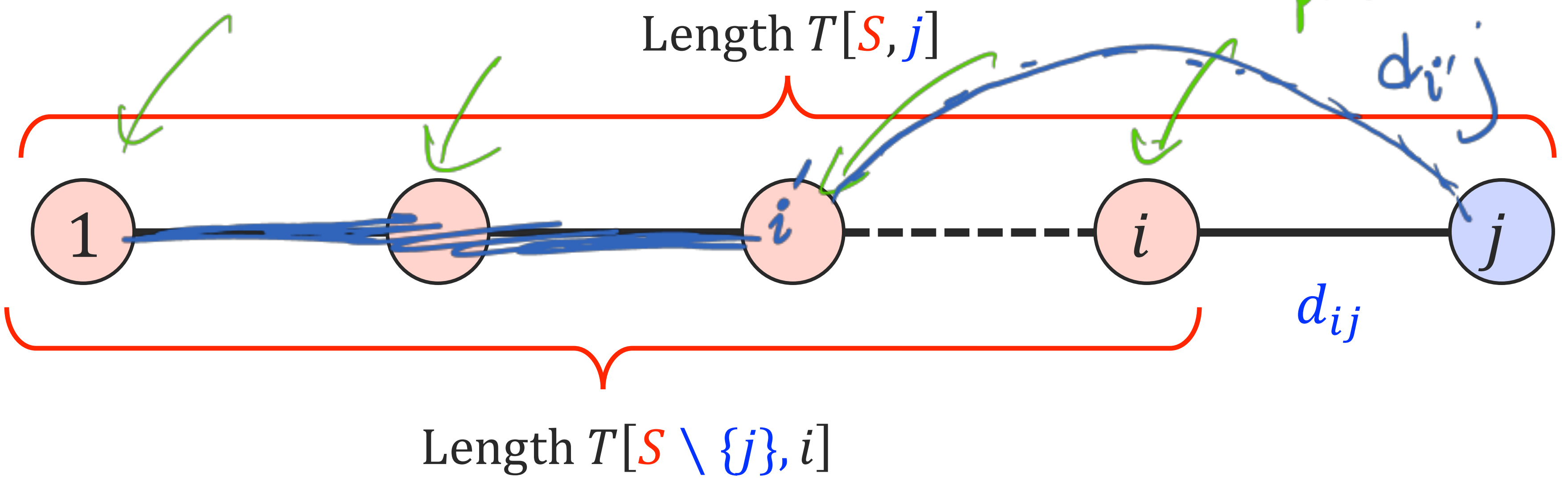
Input: cities $1 \ldots n$ and pairwise distances d_{ij} between cities i and j.

Output: A "tour" of minimum total distance.

Recurrence relation: We don't know which city *i* is the 2nd to last.

→ Take the minimum over all $i \in S$ such that $i \neq j$.

possible i is in $S \setminus j$



→ $T[S, j] = \min\{T[S \setminus \{j\}, i] + d_{ij} \mid i \in S \text{ and } i \neq j\}$

Step 2: Base Cases and the Final Solution

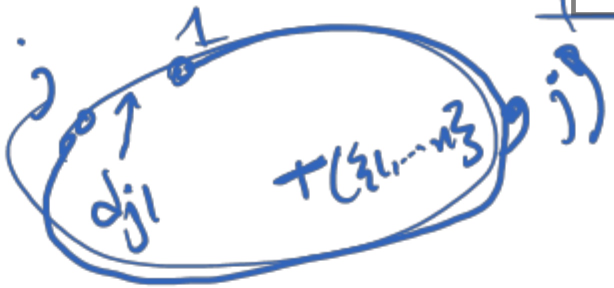
<u>Input</u>: cities $1 \dots n$ and pairwise distances d_{ij} between cities i and j.

<u>Output</u>: A "tour" of minimum total distance.

Recurrence relation: $T[S, j] = \min\{T[S \setminus \{j\}, i] + d_{ij} \mid i \in S \text{ and } i \neq j\}$

<u>Base cases</u>: $T[\{1\}, 1] = 0$ and for all other S of size ≥ 2, $\boxed{T[S, 1] = \infty}$.

No partial path allowed to start and ends at 1.



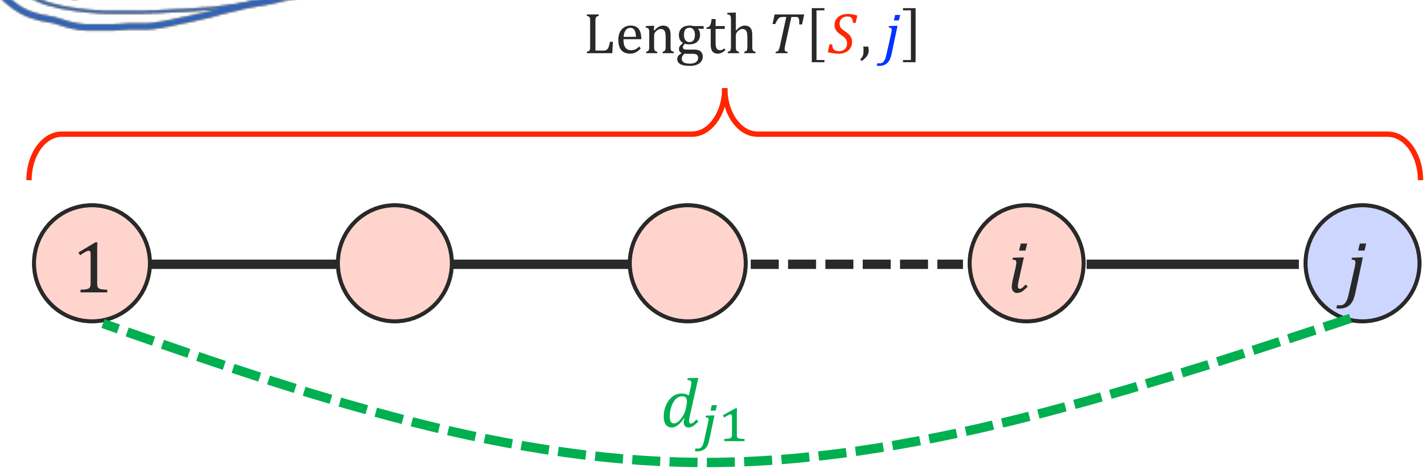
<u>Final solution:</u>

→ Add the final $(j, 1)$ edge cost:

$$T[\{1, \dots, n\}, j] + d_{j1}$$

→ Find the best j:

$$\min_{j \neq 1} T[\{1, \dots, n\}, j] + d_{j1}$$



Step 3: Design the algorithm $T[S,j]$ 2^n n

Input: cities $1 \ldots n$ and pairwise distances d_{ij} between cities i and j.

Output: A "tour" of minimum total distance.

$O(2^n \times n)$ number of subproblems.

For each subproblem, we take min of $\leq n$ values:
→ Work per subproblem $O(n)$

Total runtime: $O(n^2 2^n)$.

TSP(d_{ij}: $i, j \in [n]$)

An array T of size $2^n \times n$.

T[{1},1] = 0, T[S, 1] $= \infty$ for all sets S $S = 1, \ldots, n$

For set size $s = 2, \ldots, n$ $|S| = s$

For sets S, s.t. $|S| = s, 1 \in S$ $S \ni 1$

For $j \in S$ $S \ni j$

$$T[S, j] = \min_{i \in S:\, i \neq j} \{T[S \setminus \{j\}, i] + d_{ij}\}$$

return $\min_{j \neq 1} T[\{1, \ldots, n\}, j] + d_{j1}$