CS 170
Efficient Algorithms and Intractable Problems

Lecture 14
Dynamic Programming IV

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Announcements

Nika’s OH after class today:
→ Meet at the podium of the entrance and walk to nearby benches.
→ Submit request for 1-1 TA. Meeting by today
→ We will finish midterm regrades later this week
→ HW 7 due on Saturday

Next few weeks:
→ John Wright will be lecturing
→ I will be back for some fun lectures towards the end of the semester!

Remember him?!

Recap of the last 3 lectures

Dynamic Programming!

The recipe!

Step 1. Identify subproblems (aka optimal substructure)
Step 2. Find a recursive formulation for the subproblems
Step 3. Design the Dynamic Programming Algorithm
→ Memo-ize computation starting from smallest subproblems and building up.

We saw a lot of examples already
→ Shortest Paths (in DAGs, Bellman-Ford, and All-Pair), Longest increasing subsequence, Edit distance, Knapsack, Traveling Salesman Problem, …

This lecture

Last lecture on Dynamic Programming
→ Independent Sets on Trees
→ Chain Matrix Multiply

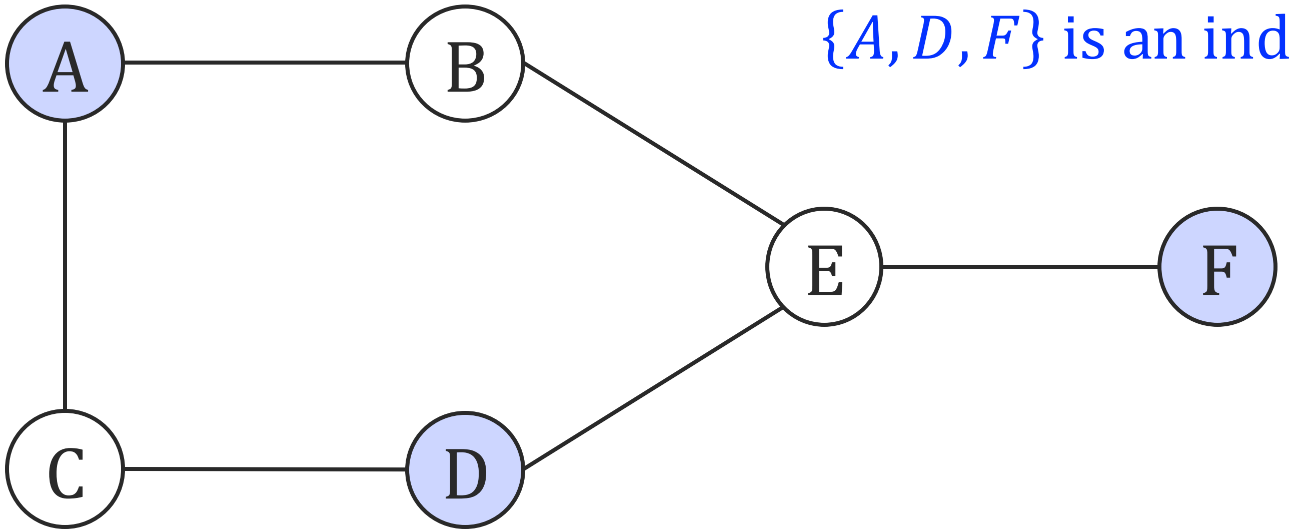
Best way to learn dynamic programming is by doing a lot of examples!

Independent Sets (in Trees)

<u>Input</u>: Undirected Graph $G = (V, E)$

<u>Output</u>: Largest "independent set" of G.

Definition: $S \subseteq V$ is an **independent set** of G if there are no edges between any $u, v \in S$.



$\{A, D, F\}$ is an independent set.

Independent Sets (in Trees)

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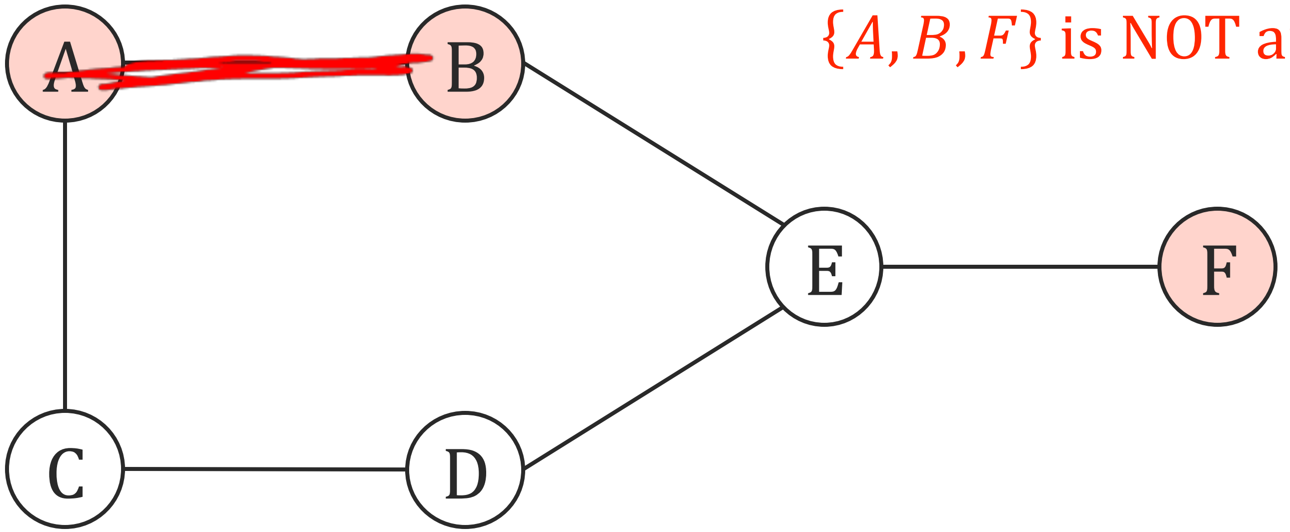


$\{A, B, F\}$ is NOT an independent set.

Finding largest independent set **can't be done in polynomial** time in **general graphs**.
For **trees**, dynamic programming gives $O(|V|)$ algorithm!

Independent Sets in Trees

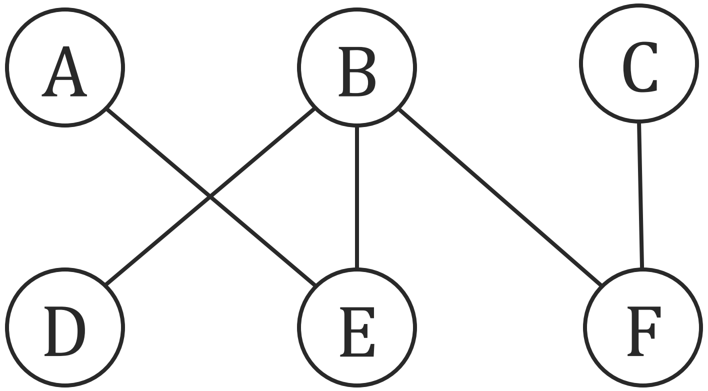
Input: Undirected Graph $G = (V, E)$ and G is a tree.

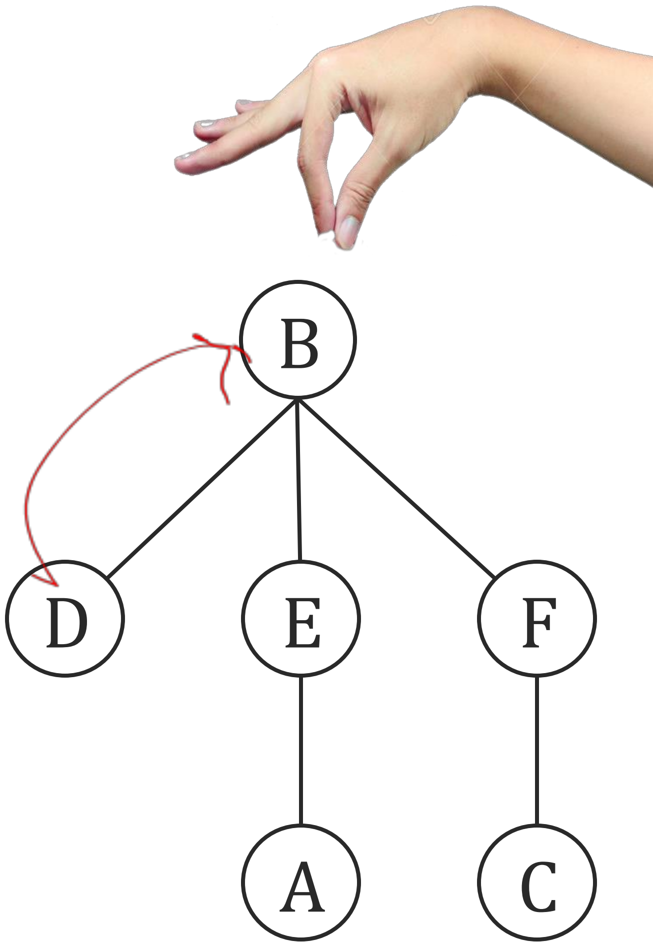
Output: Largest "independent set" of G.

Recall, trees don't have cycles!

→ We can pick and node of a tree and say that it's the **root**

→ Rooted trees create a natural order between nodes, parent to children.

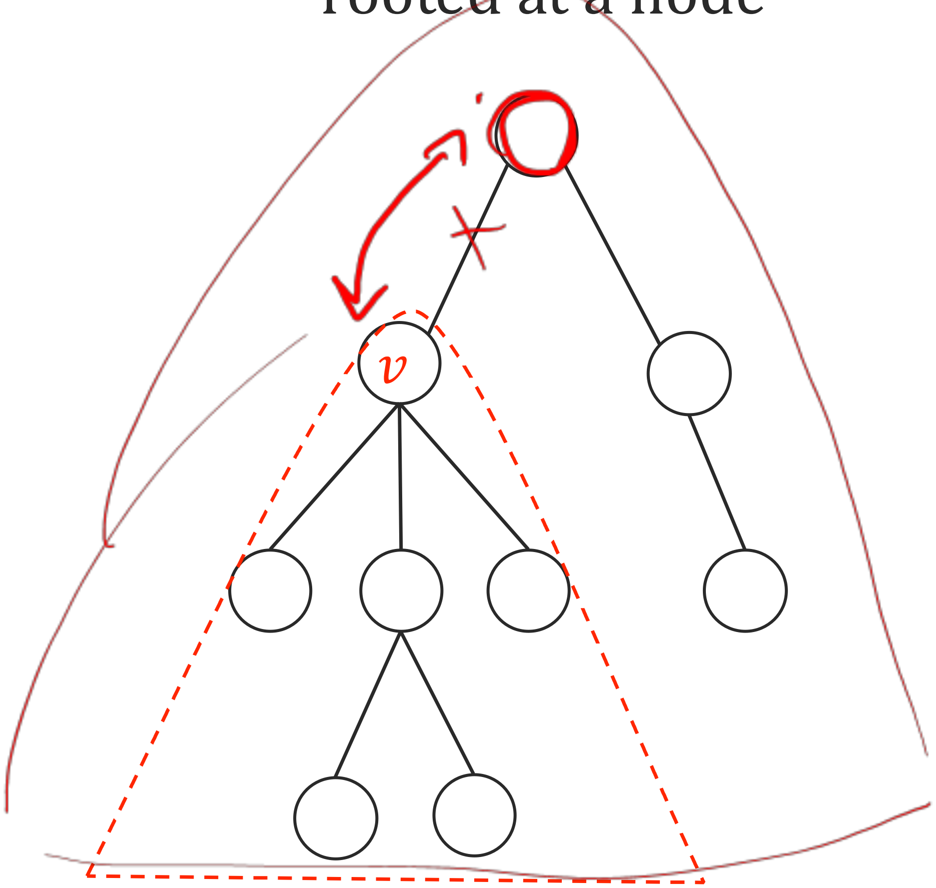




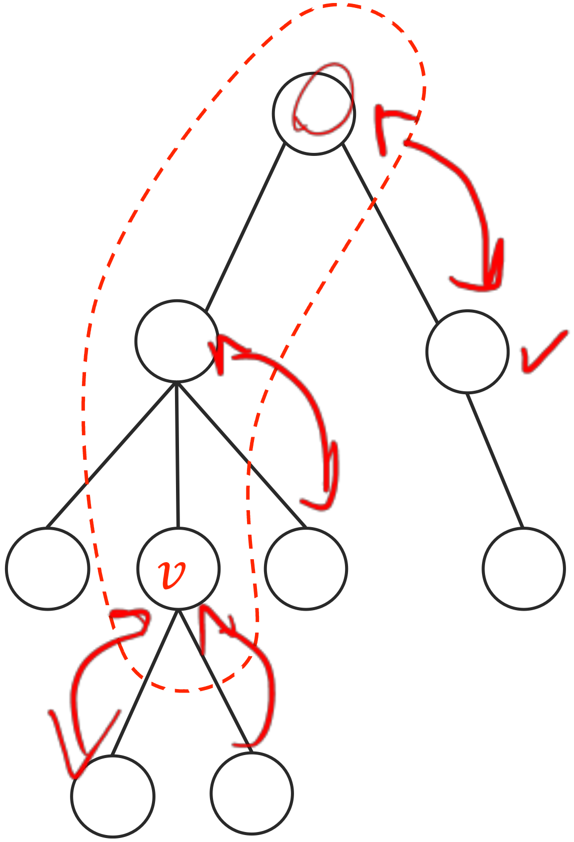
Which choice of subproblem is more appropriate?

Discuss

Max IS in the subtree rooted at a node



Max IS in the ancestors of a node



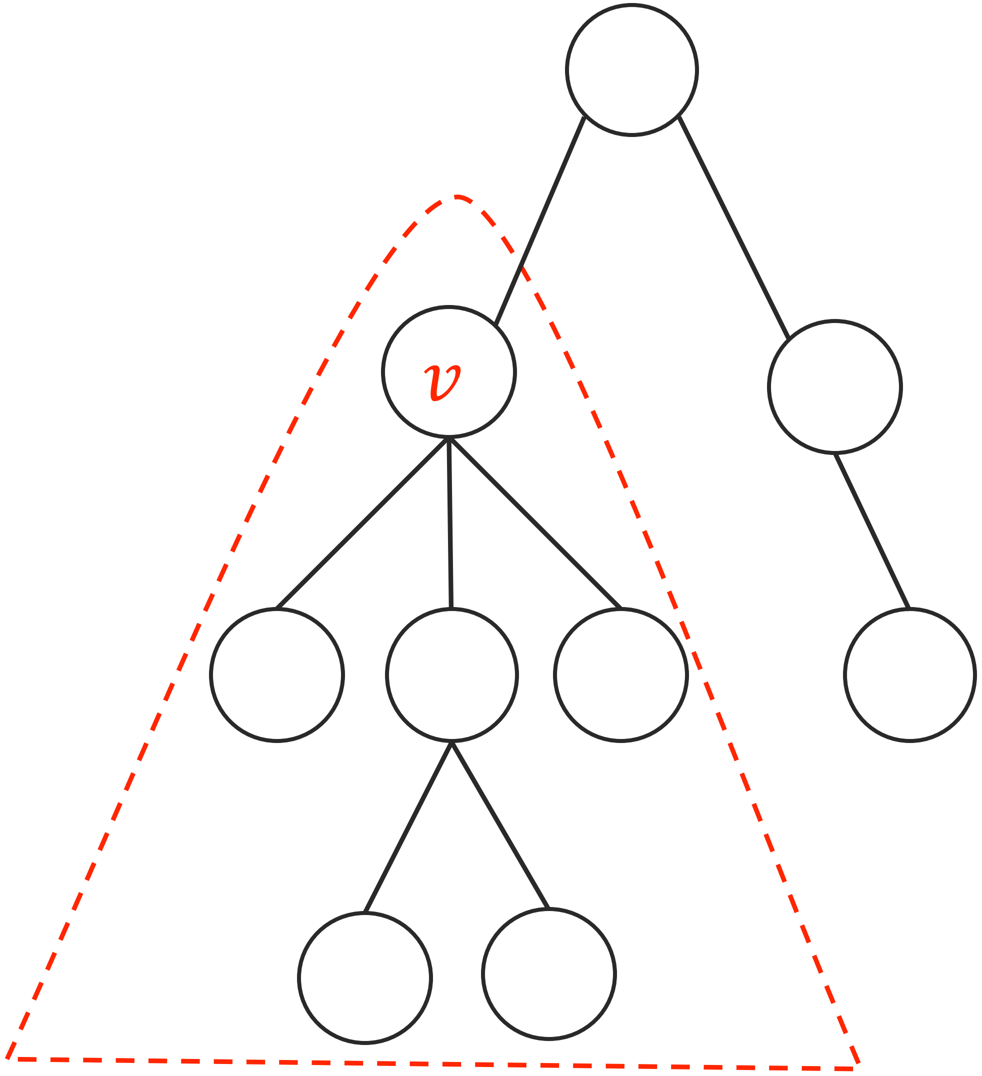
Step 1: Subproblems for Independent Sets

Input: Undirected Graph $G = (V, E)$ and G is a tree.

Output: Largest "independent set" of G.

Subproblems: For each $v \in V$

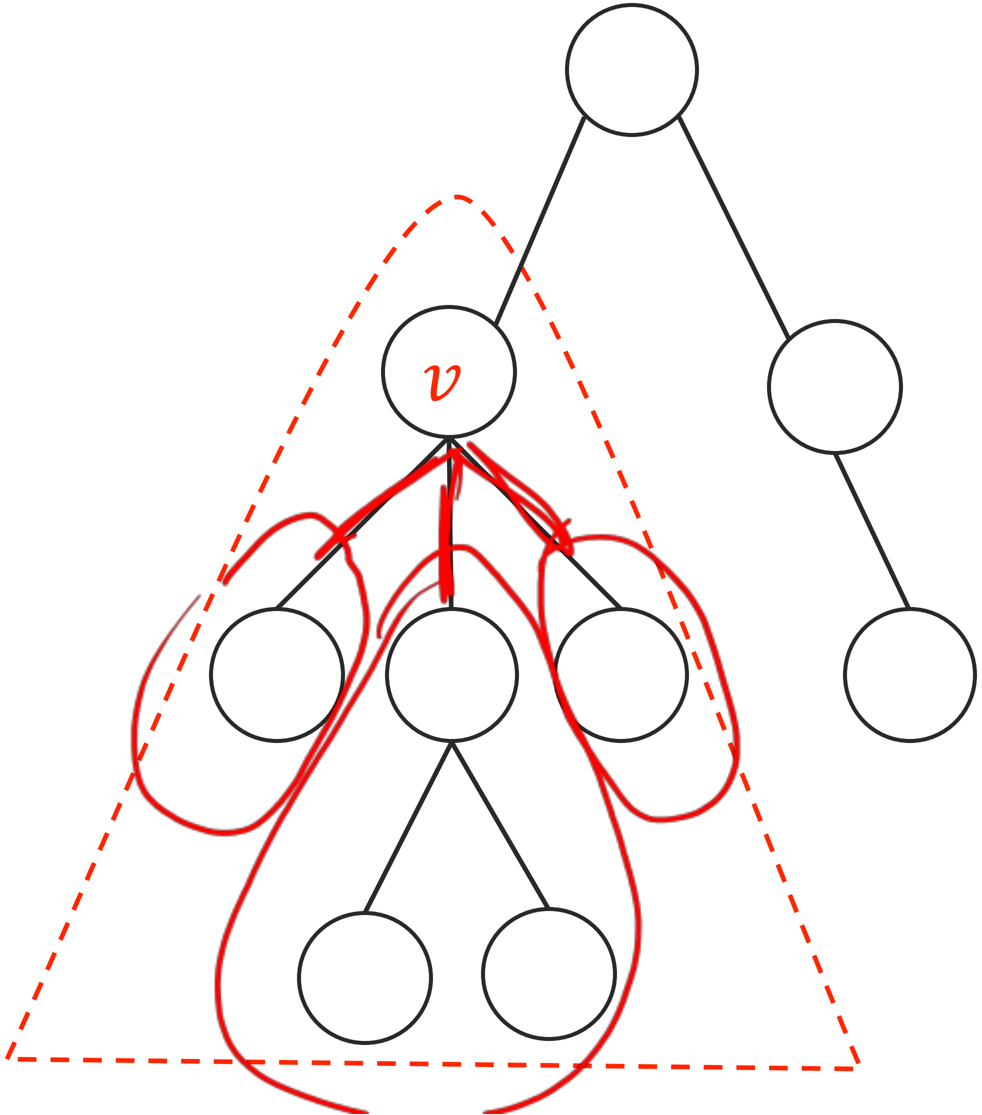
$I(v) =$ Size of max independent set in subtree rooted at v.



Step 2: Recurrence for Independent Sets

<u>Input</u>: Undirected Graph $G = (V, E)$ and G is a tree.

<u>Output</u>: Largest "independent set" of G.



Subproblems: For each $v \in V$

$I(v) =$ Size of max independent set in subtree rooted at v.

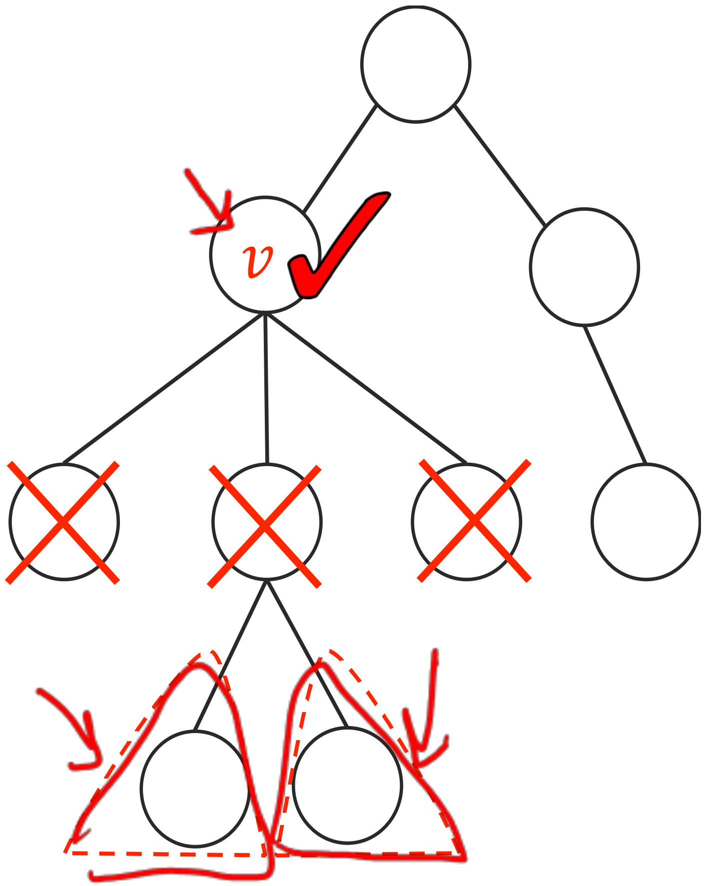
Recurrence: Compute $I[v]$ using smaller subproblems (its descendants)

Two Cases:

Max Independent Set in Subtree rooted at v.

Recurrence: Compute $I[v]$ using smaller subproblems (its descendants)

Case 1: The optimal solution for $I[v]$ uses v.



None of the children of v can be in the independent set.

Recurse to the grandchildren levels:

$$I[v] = 1 + \sum_{u:\text{grandchild of } v} I[u]$$

Two Cases:

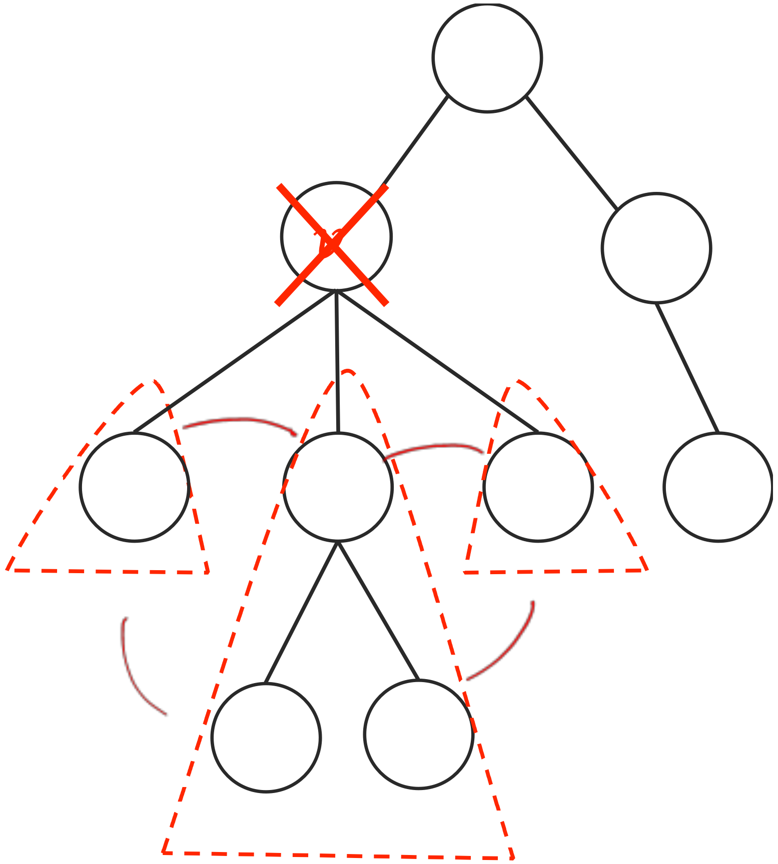
Recurrence: Compute $I[v]$ using smaller subproblems (its descendants)

Case 2: The optimal solution for $I[v]$ does NOT use v.

This doesn't restrict the optimal solution in the children of v.

Recurse to the children levels:

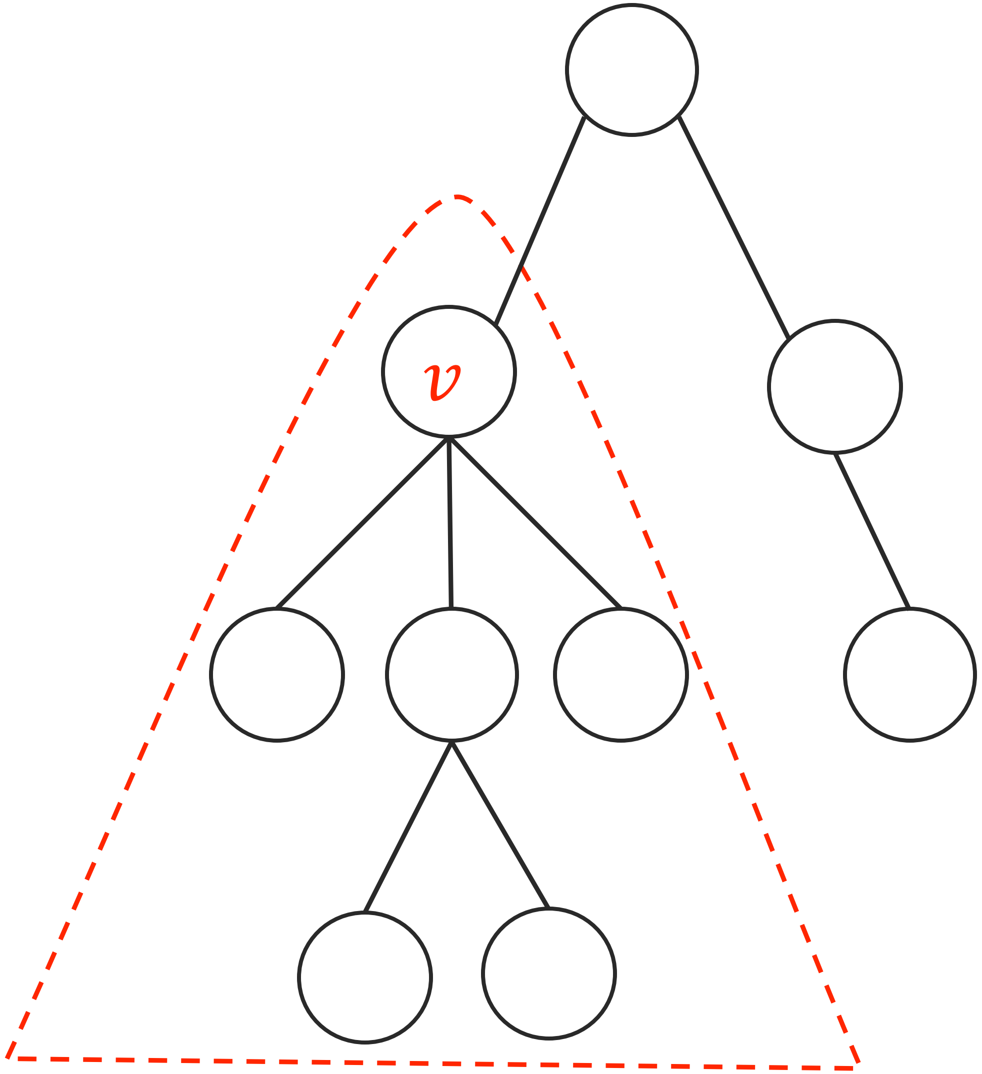
$$I[v] = \sum_{u\text{: child of } v} I[u]$$



Step 2: Recurrence for Independent Sets

<u>Input</u>: Undirected Graph $G = (V, E)$ and G is a tree.

<u>Output</u>: Largest "independent set" of G.



Subproblems: For each $v \in V$

$I(v) =$ Size of max independent set in subtree rooted at v.

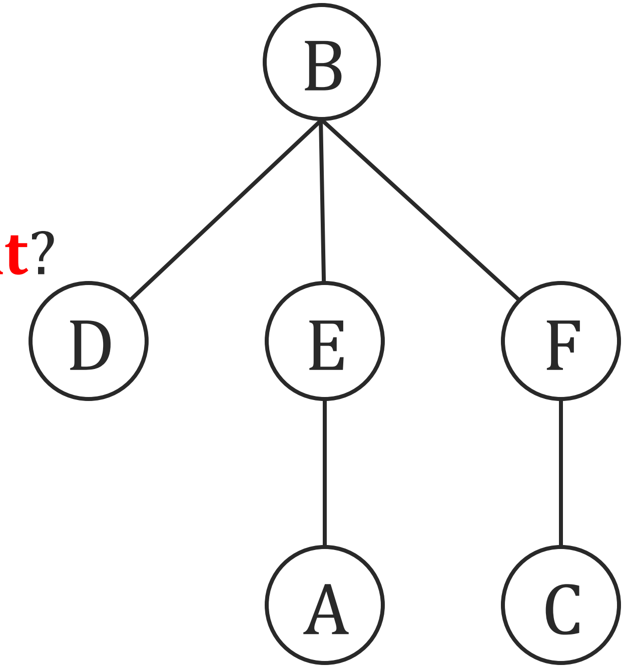
Recurrence: Compute $I[v]$ using smaller subproblems (its descendants)

$$I[v] = \max\left\{1 + \sum_{u\text{:grandchild of } v} I[u],\ \sum_{u\text{: child of } v} I[u]\right\}$$

Step 3: Design the Algorithm

Input: Undirected Graph $G = (V, E)$ and G is a tree.

Output: Largest "independent set" of G.



We need a data structure to store the tree easily.

→ How to ensure that **every child is processed before the parent**?

Lecture 5-6 material!

Recall, post numbers in DFS(G):

- If u is a descendent of v: $post(u) < post(v)$.

Bottom-up: memo-ize in **increasing order** of $post$ numbers, in any DFS traversal.

Step 3: Design the Algorithm

I[v] = Max Independent Set Size in subtree rooted at v.

<u>Input</u>: Undirected Graph $G = (V, E)$ and G is a tree.

<u>Output</u>: Largest "independent set" of G.

1. In trees: $|E| = |V| - 1$.
2. DFS Runtime $= O(|V|)$ ⇒ O(V+E)

3. Each edge is looked at $\leq$ 2 times.

→ Once for its parent's subproblem.

→ Once for its grandparent's subproblem.

Total work for all subproblems $=$ $O(|E|) = O(|V|)$.

Total runtime: $O(|V|)$.

Independent-Set-Tree($G = (V, E)$)

An array I of size n.

sort $v_1 \dots v_n$ in increasing post order of DFS(G)

For $i = 1, \dots, n$

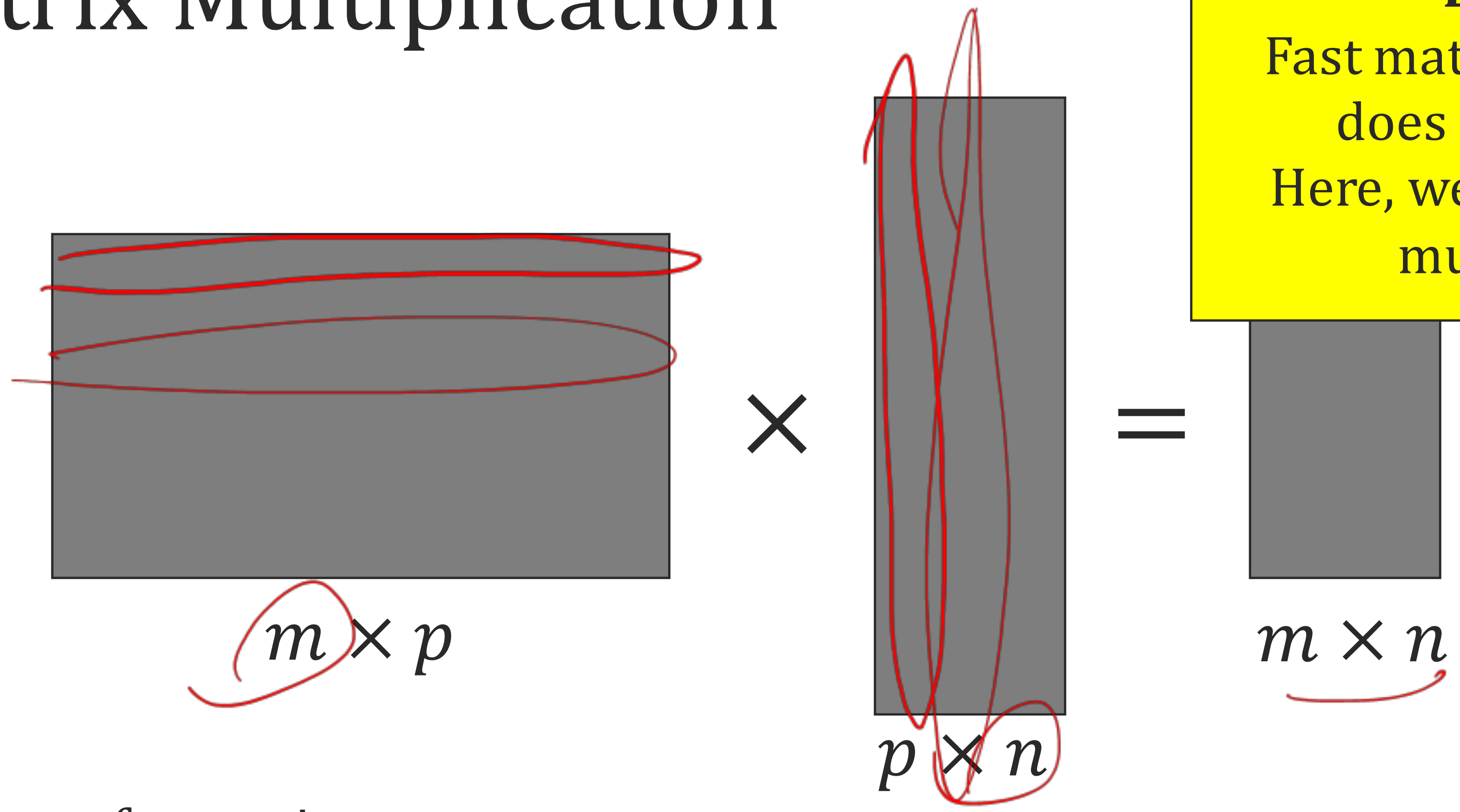
$$I[v_i] = \max \left\{ \begin{array}{l} 1 + \sum_{u:\text{grandchild of } v_i} I[u], \\ \sum_{u:\text{ child of } v_i} I[u] \end{array} \right\}$$

return $I[v_n]$

Chain Matrix Multiplication

Matrix Multiplication

Lecture 2:
Fast matrix multiplication does slightly better!
Here, we work with naïve multiplication.



Number of operations:

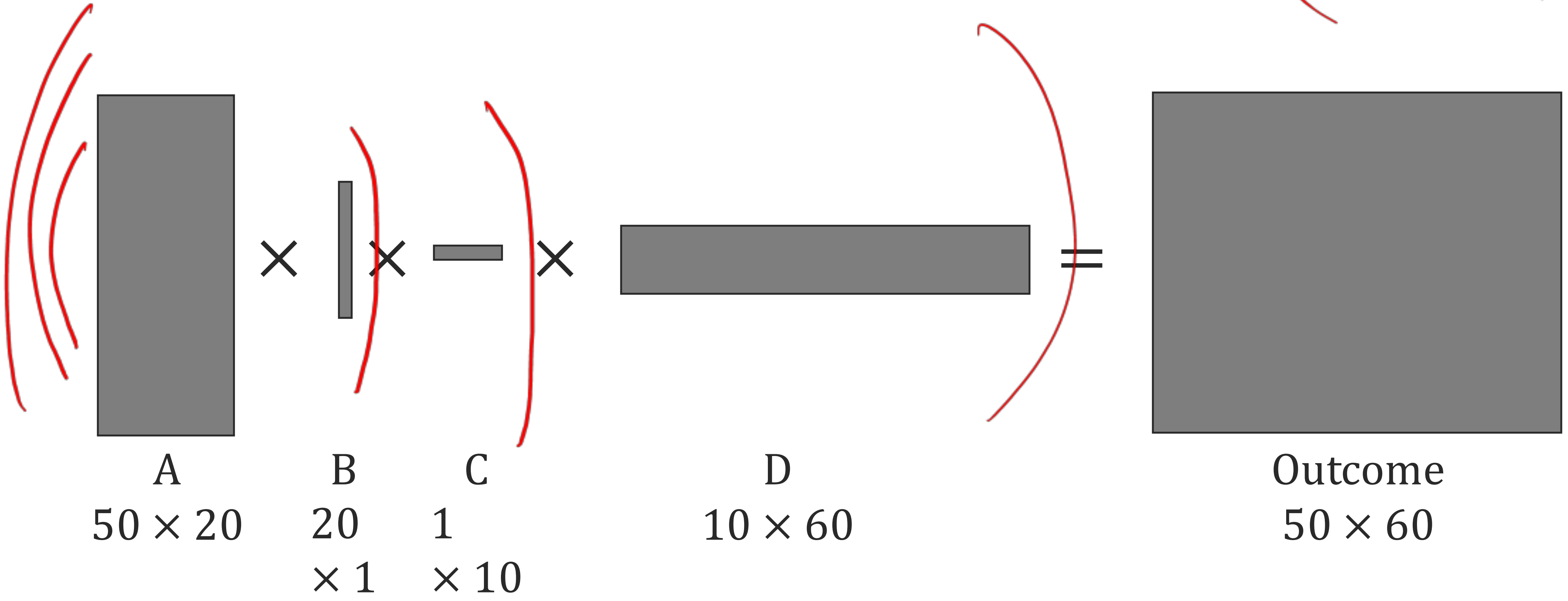
→ Outcome matrix of size $m \times n$

→ Each cell is a dot product of two vectors of length p, so $O(p)$

→ Total: $O\boxed{(mnp)}$

Chain Matrix Multiplication

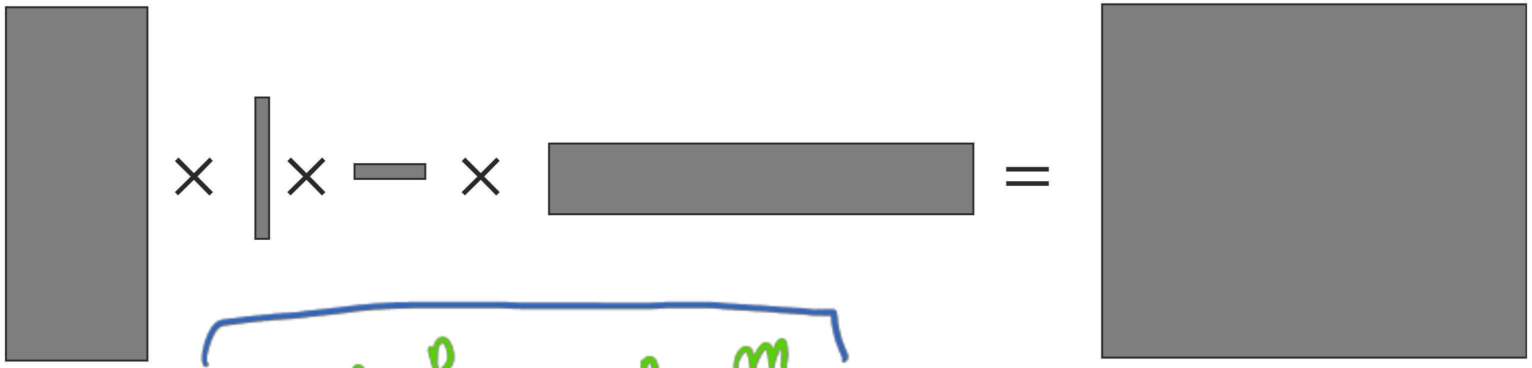
$A \times (B \times (C \times D))$



Matrix multiplication is associative (can put parenthesis anywhere), but not commutative (can't switch left and right order)

$A \times B \neq B \times A$

Chain Matrix Multiplication



A	B	C	D	Outcome
50×20	20×1	1×10	10×60	50×60

Parenthesization	Cost of Computation
$A \times ((B \times C) \times D)$	20x1x10 + 20x10 x 60 + 50 x 20x60 = 72,000
$(A \times (B \times C)) \times D$	20x1x10 + 50x20 x 10 + 50x10 x60 = 40,200
$(A \times B) \times (C \times D)$	50x20x1 + 1x10x60 + 50x1 x60 = 4,600 ?

AxB CxD

Chain Matrix Multiplication

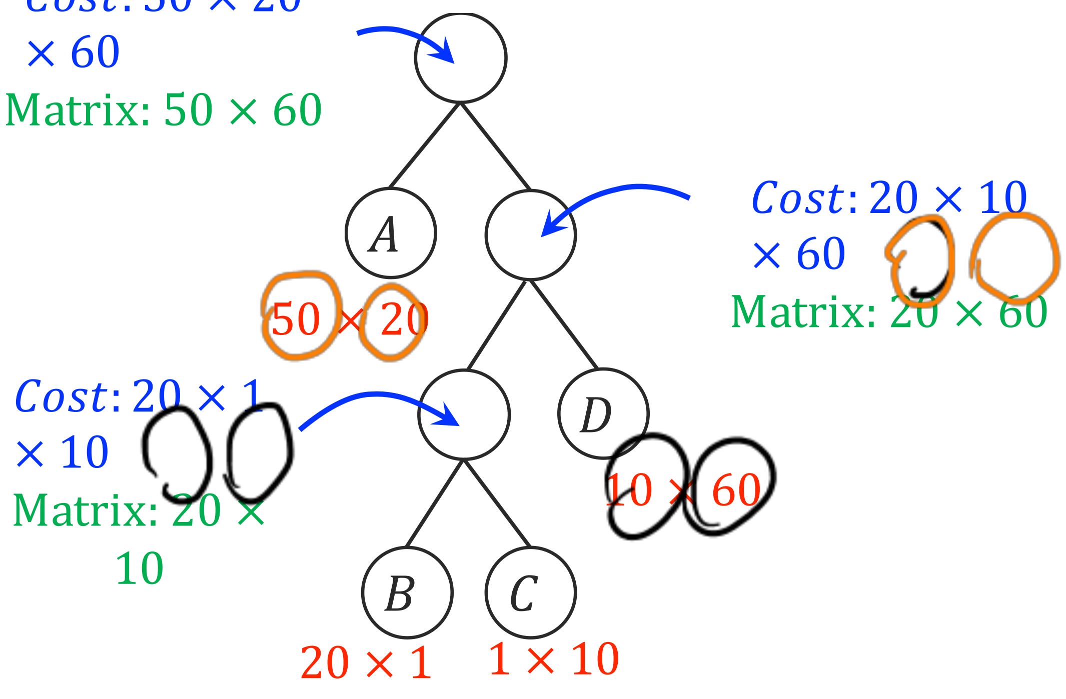
A_i A_{i+1}

Input: Matrices $A_1, \dots, A_n$, where matrix A_i is of dimension $m_{i-1} \times m_i$.

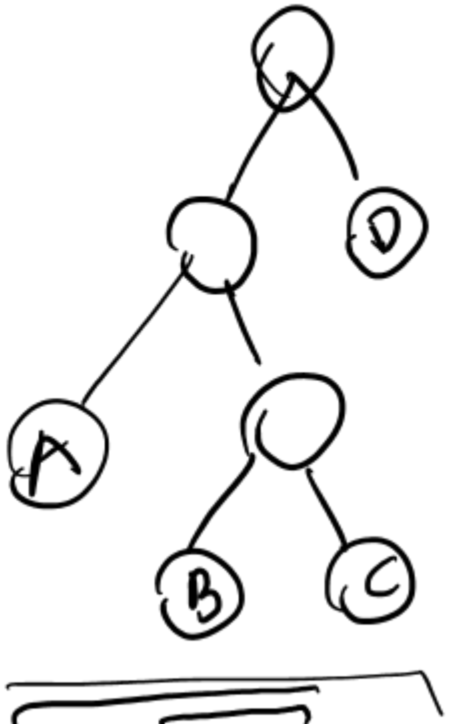
$m_i \times m_{i+1}$

Output: Minimum cost of multiplying $A_1 \times \ \dots, \times A_n$.

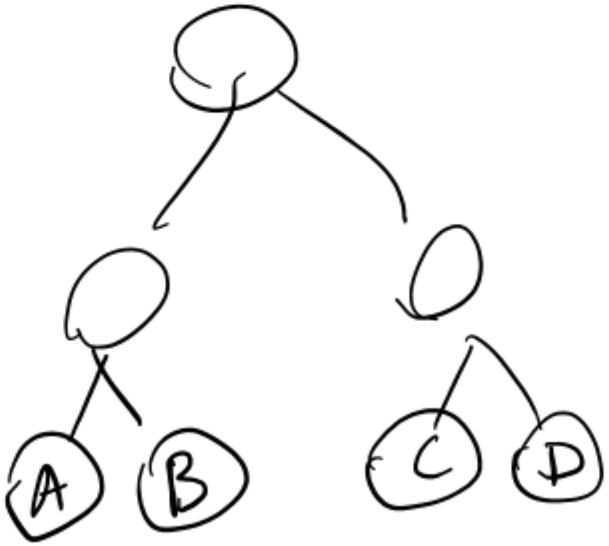
Parenthesizations correspond to binary Trees



$A \times ((B \times C) \times D)$



$(A \times (B \times C)) \times D$



$(A \times B) \times (C \times D)$

Step 1: Subproblems

<u>Input</u>: Matrices $A_1, \dots, A_n$, where matrix A_i is of dimension $m_{i-1} \times m_i$.
<u>Output</u>: Minimum cost of multiplying $A_1 \times \ \dots, \times A_n$.

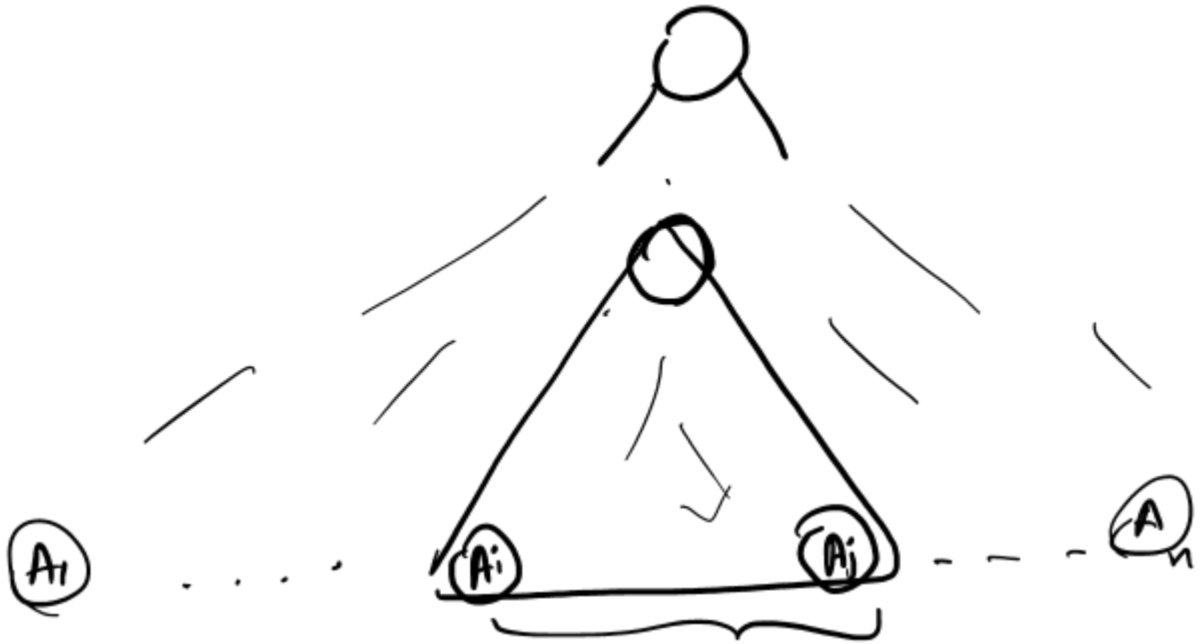
Subproblem choice: The cost of multiplying a contagious subset of the matrices

$Cost[i, j]$ =Minimum cost of multiplying $A_i \times A_{i+1} \dots, \times A_j$ for $i \leq j$

Why is this a good choice?

For a tree to be optimal, every subtree also has to be optimal.

Natural subproblem order, start from leaves and consider every subtree.



Step 2: Recurrence Relation

<u>Input</u>: Matrices $A_1, \dots, A_n$, where matrix A_i is of dimension $m_{i-1} \times m_i$.
<u>Output</u>: Minimum cost of multiplying $A_1 \times \dots, \times A_n$.

Subproblem choice: The cost of multiplying a contagious subset of the matrices

$$Cost[i,j] = \text{Minimum cost of multiplying } A_i \times A_{i+1} \dots, \times A_j \text{ for } i \le j$$

$m_{i-1} \times m_i \cdots\cdots m_{k-1} \times m_k$

To multiply $A_i \times A_{i+1} \dots, \times A_j$, we have to parenthesize it, say by splitting at k:

$m_k \times m_{k+1} \times \cdots\cdots m_{j-1} \times m_j$

$$A_i \times A_{i+1} \dots, \times A_j = (A_i \times \cdots \times A_k) \times (A_{k+1} \times \cdots A_j):$$

$$Cost[i,j] = Cost[i,k] + Cost[k+1,\cancel{n}\,j] + \text{Cost of multiplying } m_{i-1} \times m_k \text{ by } m_k \times m_j \text{ matrices}$$

$$= Cost[i,k] + Cost[k+1,\cancel{n}\,j] + m_{i-1} \times m_k \times m_j$$

For the best parenthesization of the $A_i \times A_{i+1} \dots, \times A_j$:

$$Cost[i,j] = \min_{k: i \le k \le j} \{Cost[i,k] + Cost[k+1,\cancel{n}\,j] + m_{i-1} \times m_k \times m_j\}$$

Order of Computation

$A_1 \cdots (A_i \cdots A_j) \cdots A_n$

$$Cost[i,j] = \min_{k:i\le k\le j}\{Cost[i,k] + Cost[k+1,n] + m_{i-1}\times m_k \times m_j\}$$

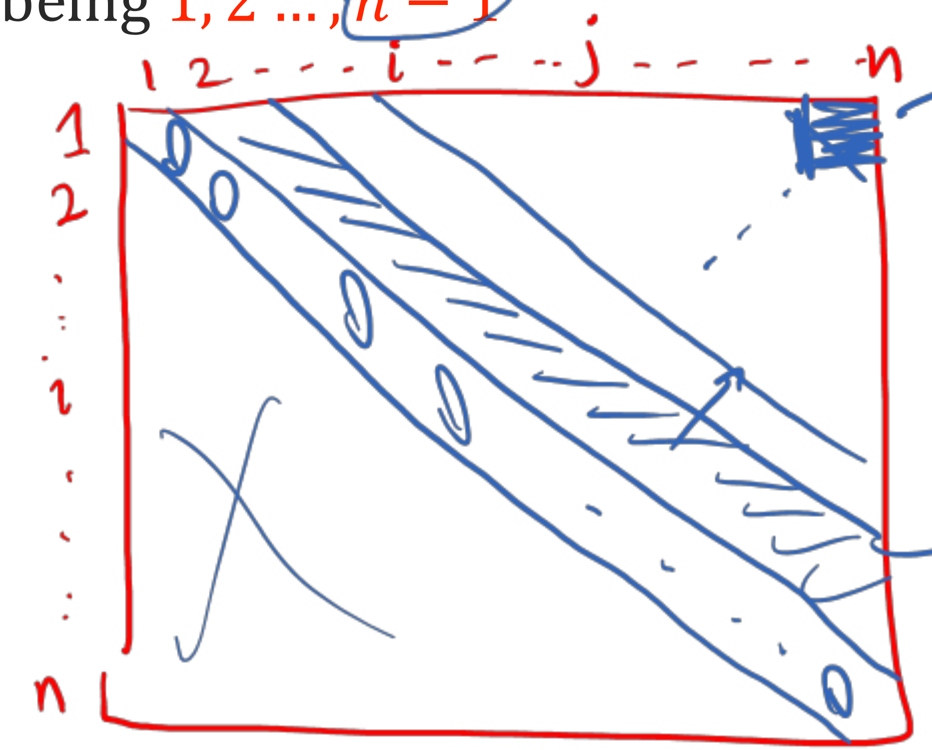
Go by the increasing size of $j-i$:

→ Base case: $Cost[i,i] = 0$ for all $i = 1,\dots,n$

→ Start from $s = j - i$ being $1, 2 \dots, n-1$

→ Fill in diagonally

Cost(i,j)



Cost(1,n)

$\Downarrow$

$A_1 \cdots\cdots A_n$

$j-i=2$

$j-i=1$

Step 3: Memo-ization

<u>Input</u>: Matrices $A_1, \ldots, A_n$, where matrix A_i is of dimension $m_{i-1} \times m_i$.
<u>Output</u>: Minimum cost of multiplying $A_1 \times \ldots, \times A_n$.

Number of subproblems is $O(n^2)$

Per subproblem:

- Minimize over $O(n)$ choices for identity of k.
- Each value takes $O(1)$ to compute

→ Total of $O(n)$ cost per subproblem.

Total runtime $O(n^3)$

Chain-Matrix-Mult($m_0, m_1, \cdots, m_n$)

cost

An array C of size $n \times n$

For $i = 1, \ldots, n$, $C[i, i] = 0$

For $s = 1 \ldots, n-1$

For $i = 1, \ldots, n - s$

$j \leftarrow i + s$

$$C[i,j] = \min_{k: i \le k < j} \begin{Bmatrix} Cost[i,k] + Cost[k+1, \cancel{n} j] \\ + m_{i-1} \times m_k \times m_j \end{Bmatrix}$$

Return $C[1, n]$

↳ $A_1 \times - - \times A_n$

Summary of Subproblem

Remember the Recipe

The recipe!

Step 1. Identify subproblems (aka optimal substructure)
Step 2. Find a recursive formulation for the subproblems
Step 3. Design the Dynamic Programming Algorithm
→ Memo-ize computation starting from smallest subproblems and building up.

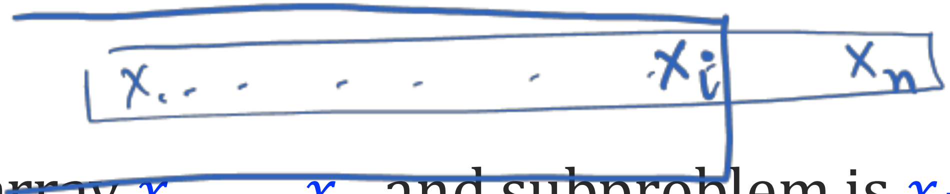
What makes for good subproblems?

- Not too many of them (the more subproblems the slower the DP algorithm)
- Must have enough information in it to compute subproblems recursively (needed for step 2).

Common Subproblem on Arrays

The input is an array $x_1, \dots, x_n$ and subproblem is $x_1, \dots, x_i$

$O(n)$ subproblems

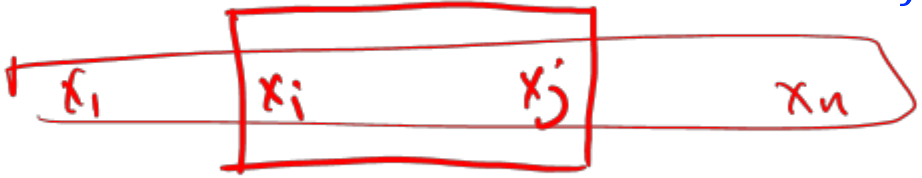


Longest Inc Subseq

DAG

The input is an array $x_1, \dots, x_n$ and subproblem is $x_i, \dots, x_j$

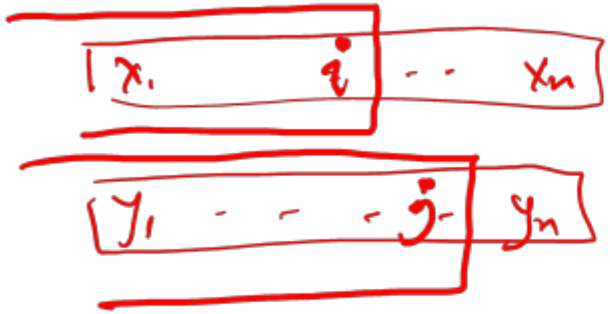
$O(n^2)$ subproblems



Chain

→ Matrix Multiply

The input is two array $x_1, \dots, x_n$ and $y_1, \dots, y_n$ and subproblems $x_1, \dots, x_i$ and $y_1, \dots, y_j$ or in some cases $x_i, \dots, x_j$ and $y_r, \dots, y_s$.

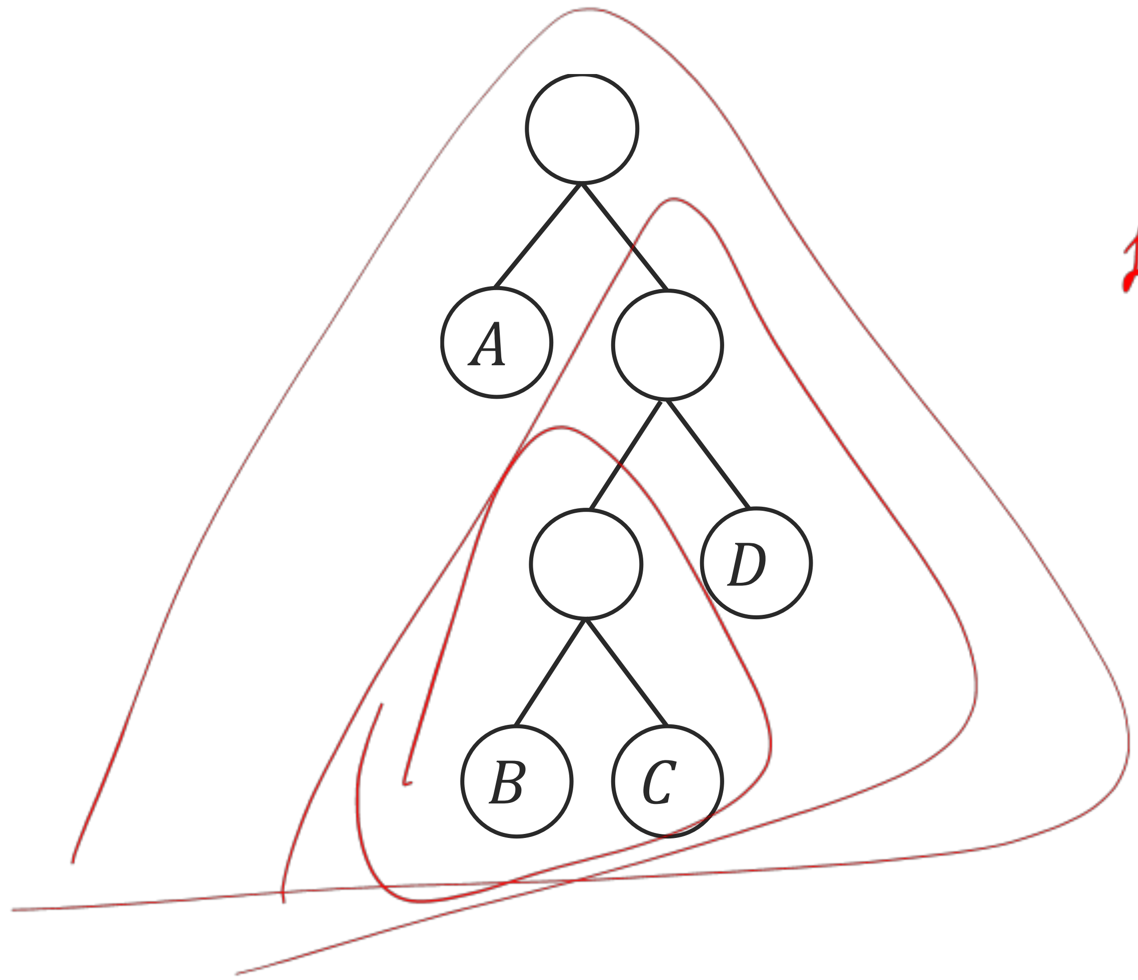


Edit distance

?

Common Subproblems on Trees

The input is a tree (or something that can be interpreted as a tree), the subproblems are subtrees



Independent Set

Common Subproblems for Graphs

You might need more creativity!

Problem might be about cycles (like Traveling salesperson), but it's easier to think about subpaths as subproblems:
→ It is harder to recurse from a big cycle to a smaller cycles
→ It is easier to recurse from a longer path to a shorter path

Problem might be about paths (like All-Pair Shortest Path, or TSP), but it helps to track internal vertices:
→ Subproblems may need to take into account sets of vertices
→ Sets like $\{x_1, \dots, x_j\}$ for all j (e.g., Floyd Warshall) or all subsets of $\{x_1, \dots, x_n\}$ (e.g., Traveling Saleperson).

Wrap up

We did lots of dynamic programming!

Dynamic programming can be best learned by practice! Do lots more example at home.

Next time:

→ Linear Programming