Linear Programming

John's bakery

Menu: Donuts $5
Cakes $25

Recipes

Ingredients	Donuts	Cake
Flour (200 total)	2	5
Sugar (300 total)	2	9
Eggs (500 total)	7	12

Goal: How many cakes & donuts to make to maximize profit?

Linear program (LP)

Decision variables:

x = # of donuts
y = # of cakes
(in $\mathbb{R}$)

Constraints: (Linear)

$x \geq 0, \quad y \geq 0$

$2x + 5y \leq 200$

$2x + 9y \leq 300$

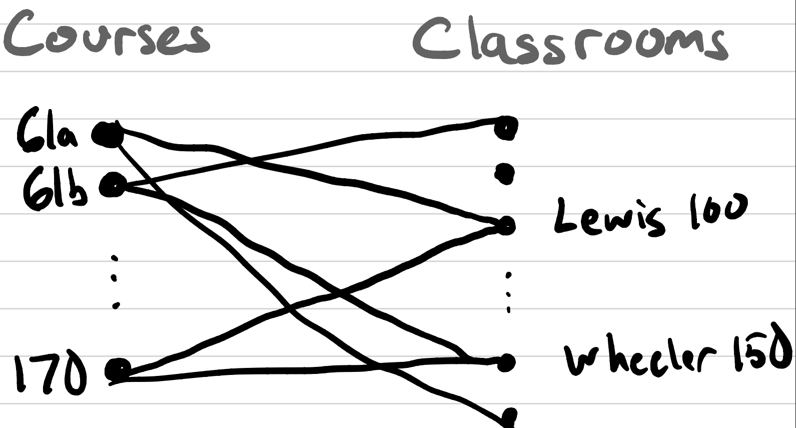
$7x + 12y \leq 500$

x and y are integers

Not allowed!
Gives Integer LP

Objective: maximize $5x + 25y$
(Linear)

Classroom allocation



Class c can be assigned room r

$\Longleftrightarrow$

$(c, r) \in E$

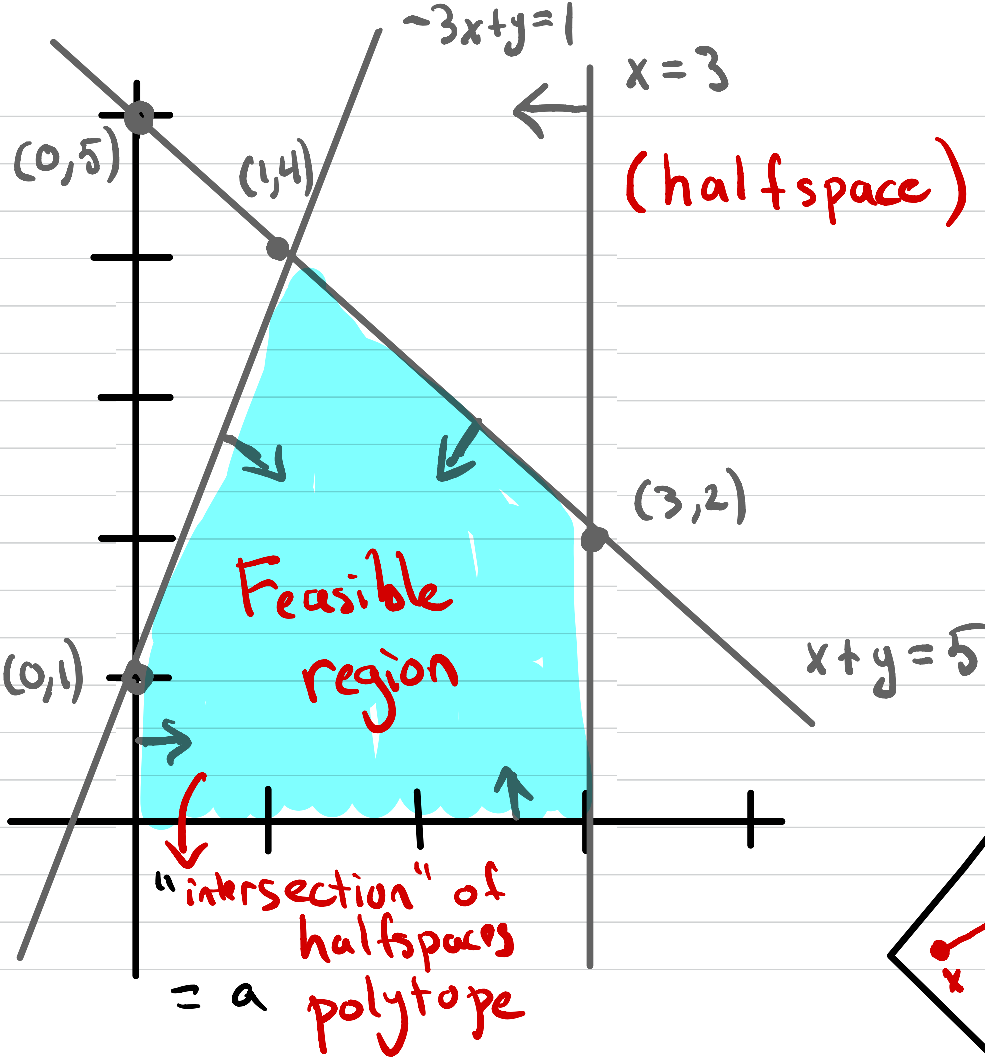
Goal: Maximize number of courses assigned classrooms

Linear Program (LP)

Variables

Constraints

Objective:



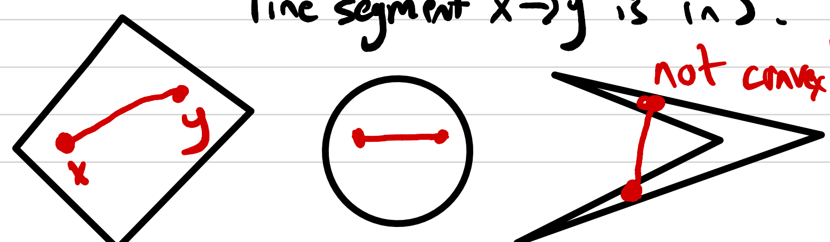
LP: max $x+2y$

subject to
$$\begin{aligned} x &\le 3 \\ x+y &\le 5 \\ -3x+y &\le 1 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

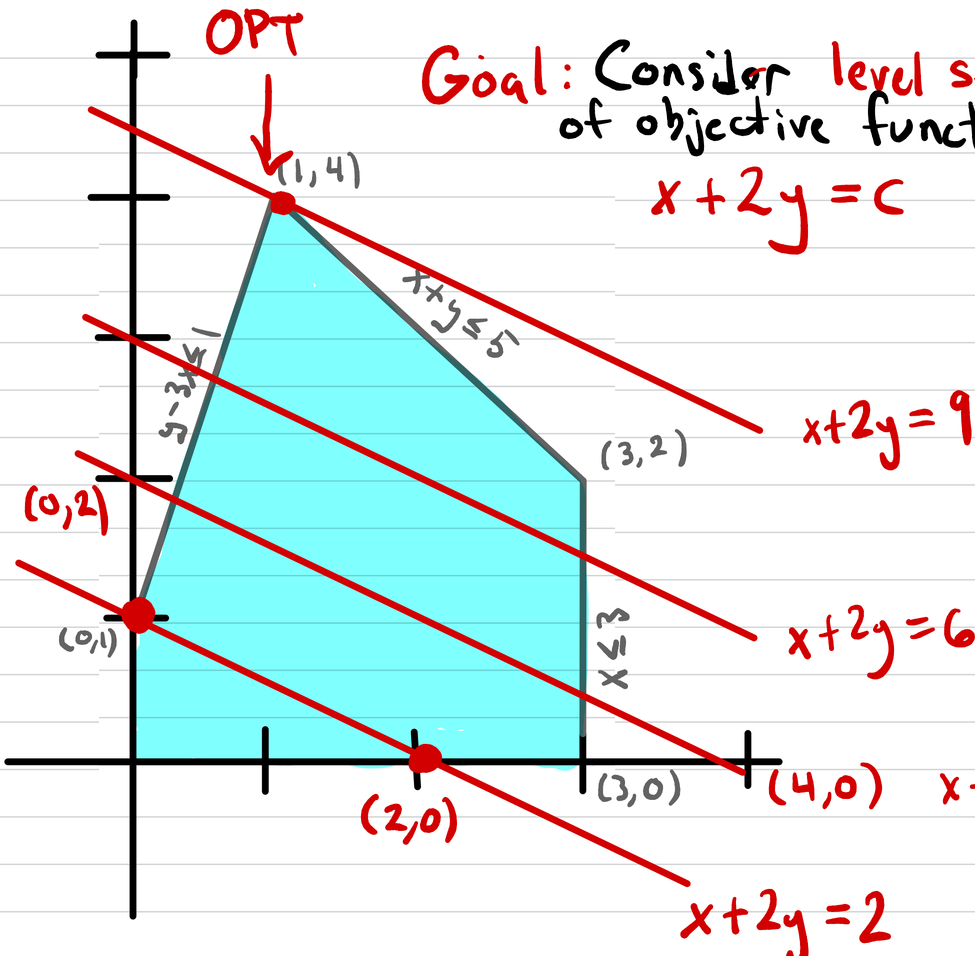
Def: The feasible region is the set of points satisfying all constraints

Fact: The feasible region is convex

Def: A set of points $S \subseteq \mathbb{R}^d$ is convex if $\forall\, x, y \in S$, line segment $x \to y$ is in S.



Goal: Consider level sets of objective function $x+2y=c$



LP: max $x+2y$

subject to
$$x \le 3$$
$$x+y \le 5$$
$$-3x+y \le 1$$
$$x \ge 0$$
$$y \ge 0$$

Fact: For all LPs, there is an optimal solution at a vertex

Fact: Can prove $OPT = 9$.

Pf: Any feasible point satisfies

$$\begin{array}{rl} & (x+y \le 5)\cdot 7/4 \\ + & (-3x+y \le 1)\cdot 1/4 \\ \hline & x+2y \le 9. \end{array} \qquad \square$$

"LP duality"

General LP

Input: a_{ij}'s, b_i's, c_i's

Variables:

$x_1, \ldots, x_n$

Constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_n \end{array}\right\} m$$

Objective

Maximize $c_1x_1 + \cdots + c_nx_n$

Variants

$$\sum_i a_i x_i \ge b_i \rightarrow \sum_i -a_i x_i \le -b_i$$

$$\sum_i a_i x_i = b_i \rightarrow \left\{\begin{array}{l} \sum_i a_i x_i \le b_i \\ \sum_i a_i x_i \ge b_i \end{array}\right.$$

Minimize $\sum_i c_i x_i$

$\rightarrow$ Maximize $\sum_i -c_i x_i$

General LP

Variables

$$x_1, \dots, x_n \in \mathbb{R}$$

Constraints

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m
\end{aligned}\right\} m$$

Objective

Maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Matrix form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

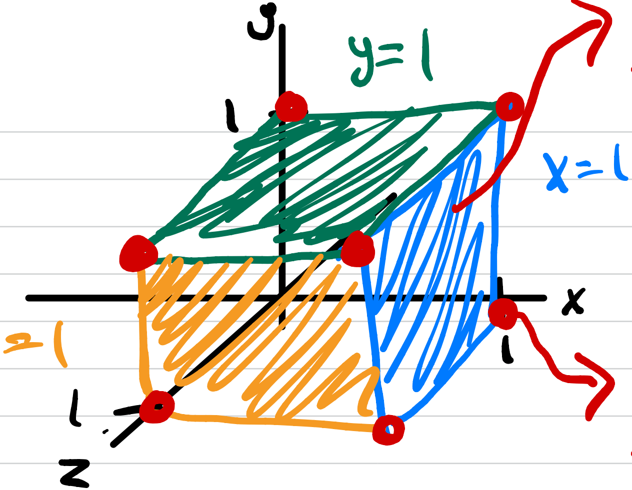
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

LP = maximize $c^T x$

subject to $Ax \le b$

LP example

$$\begin{aligned} \max \quad & x+y+z \\ \text{s.t.} \quad & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ & 0 \le z \le 1 \end{aligned}$$



in 3D: plane
in general dim: hyperplane

intersection of $x=1, y=0, z=0$

Def: A vertex is a point x in the polytope ($x \in$ feasible set) which lies at the intersection of n hyperplanes in n dimensions

Recall: Always an optimal solution at a vertex

Fact: Suppose LP has n vars and m constraints.

Then # of vertices $\le$ ________.

Linear Programming Alg 1: "Try all vertices"

Given m constraints $C = \{ \sum a_{ij} x_j \leq b_j \}$

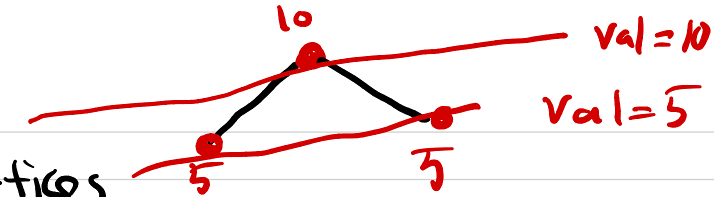
For each subset $S \subseteq C$ of size n:

1. Solve for the point of intersection x^* (solving linear system / Gaussian elimination)
2. Check if x^* is feasible (satisfies all constraints)
3. If so, compute value of x^*

Output best vertex found.

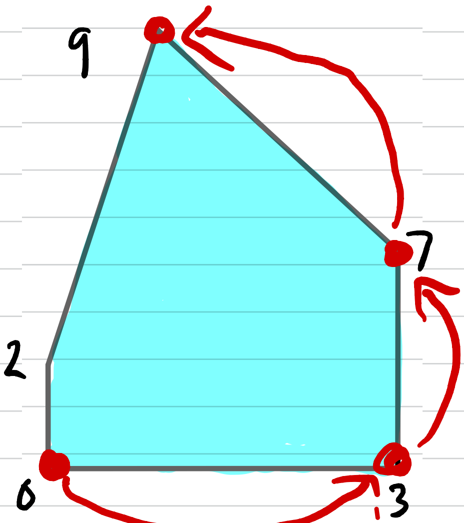
Time complexity $\approx \binom{m}{n} \approx$ exponentially large in n

LP alg 2: Simplex



1. Start at a vertex x^*
2. Look at all "neighboring" vertices
3. Find the neighbor y^* with best value
4. If y^* has a better val than x^*, move to y^*. Repeat.

$n=2$:



$n=3$: Constraints $\{A, B, C, D, E\}$

Vertex: (ABC)

Neighbors: ABD, ABE, ACD, ACE, BCD, BCE

Fact: Suppose n vars, m constraints
# of neighbors ≤ ______

Sad fact: Simplex still exponential time
(in the worst case)

Thm: Linear programs can be solved in polynomial time.

[Khachiyan 1979]: Ellipsoid algorithm

[Karmarkar 1984]: Interior points algorithm