Linear

Programming

John bakery

Menu: Donuts $5
Cakes $25

Recipes

Ingredients	Donuts	Cake
Flour (200 total)	2	5
Sugar (300 total)	2	9
Eggs (500 total)	7	12

Goal: How many cakes & donuts to make to maximize profit?

Linear program (LP)

Decision variables:

x = # of donuts
y = # of cakes
(in $\mathbb{R}$)

Constraints: (Linear)

$x \geq 0$, $y \geq 0$

$2x + 5y \leq 200$

$2x + 9y \leq 300$

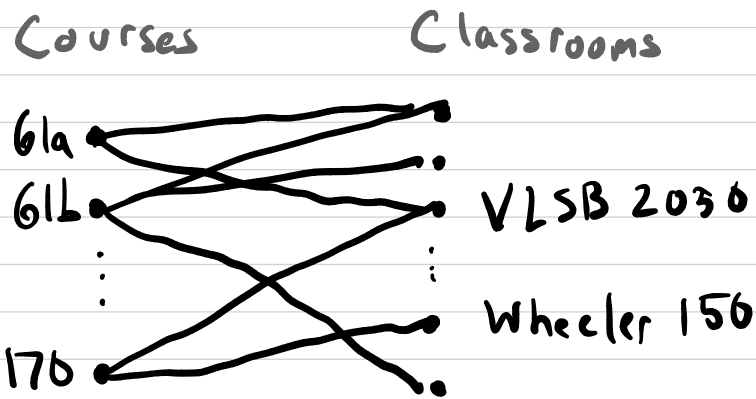
$7x + 12y \leq 500$

~~x and y are integers~~

Not allowed! Gives Integer LP

Objective: maximize $5x + 25y$
(Linear)

Classroom allocation



Class c can be assigned room r

$\Longleftrightarrow$

$(c,r) \in E$

Goal: Maximize number of courses assigned classrooms

Linear Program (LP)

Variables

$$y_c = \sum_r x_{c,r}$$

$x_{c,r}$ for each $(c,r) \in E$ $\;\in \mathbb{R}$

Constraints

for all c, r, $x_{c,r} \geq 0$

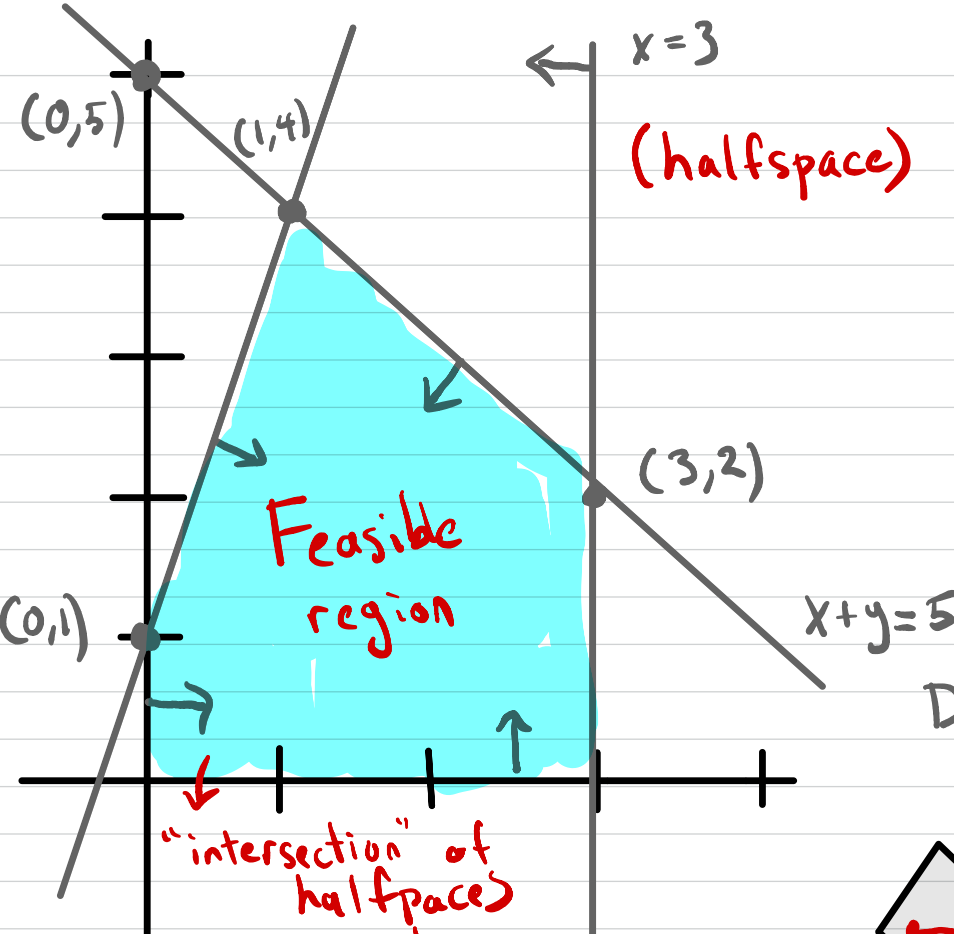
for all courses c:

$$y_c = \sum_{r:(c,r)\in E} x_{c,r} \leq 1$$

for all rooms r:

$$\sum_{c:(c,r)\in E} x_{c,r} \leq 1$$

Objective maximize $\displaystyle\sum_{(c,r)\in E} x_{c,r}$



"intersection" of halfspaces

= a polytope

LP: max $x+2y$

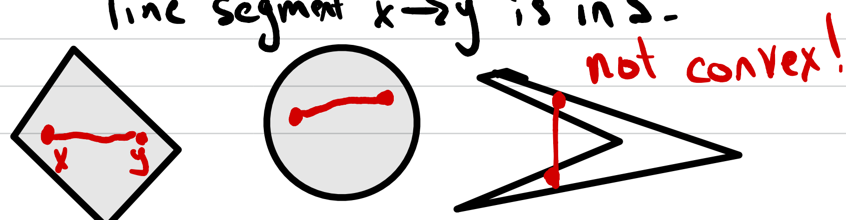
subject to

$$x \le 3$$
$$x + y \le 5$$
$$-3x + y \le 1$$
$$x \ge 0$$
$$y \ge 0$$

Def: The feasible region is the set of points satisfying all constraints

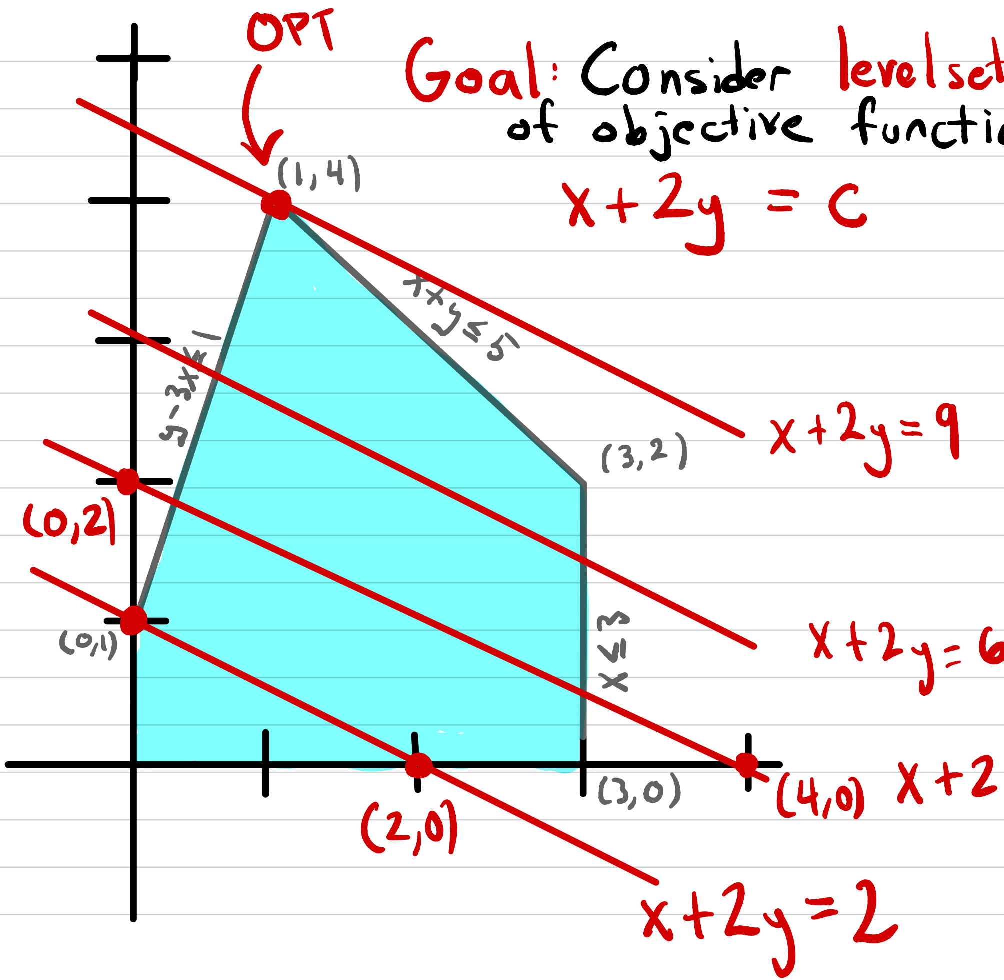
Fact: The feasible region is convex

Def: A set of points $S \subseteq \mathbb{R}^d$ is convex if $\forall\, x, y \in S$, line segment $x \to y$ is in S.



Goal: Consider level sets of objective function

$$x + 2y = c$$



LP: max $x + 2y$

subject to

$$\begin{aligned} x &\le 3 \\ x + y &\le 5 \\ -3x + y &\le 1 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

Fact: For all LPs, there is an optimal solution at a vertex

Fact: Can prove OPT = 9.

Pf: Any feasible point satisfy

$$\begin{array}{rl} & (x + y \le 5)\cdot 7/4 \\ + & (-3x + y \le 1)\cdot 1/4 \\ \hline & x + 2y \le 9 \end{array} \qquad \square$$

"LP duality"

General LP

Variables

$x_1, \dots, x_n$

Constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$

m (usually)

$$x_1 \ge 0, \quad \dots, \quad x_n \ge 0$$

m (today)

Objective

Maximize $c_1x_1 + \cdots + c_nx_n$

Input: a_{ij}'s, b_i's, c_i's

Output: x_i's

Variants

$$\sum_i a_i x_i \ge b \longrightarrow \sum_i -a_i x_i \le -b$$

$$\sum_i a_i x_i = b \longrightarrow \begin{cases} \sum_i a_i x_i \le b \\ \sum_i a_i x_i \ge b \end{cases}$$

Minimize $\sum_i c_i x_i$

$\longrightarrow$ Maximize $\sum_i -c_i x_i$

General LP

Variables

$x_1, \dots, x_n$

Input: a_{ij}'s, b_i's, c_i's

Output: x_i's

Constraints

$$\left.\begin{array}{l} \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2 \\ \vdots \end{array}\right\} m \text{ (usually)} \\ x_1 \geq 0, \dots, x_n \geq 0 \end{array}\right\} m \text{ (today)}$$

Objective

Maximize $c_1x_1 + \dots + c_nx_n$

Matrix form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \end{bmatrix}$$

LP = maximize $c^T x$

subject to $Ax \le b$

$x \geq 0$

General LP

Variables

$x_1, \ldots, x_n \in \mathbb{R}$

Constraints

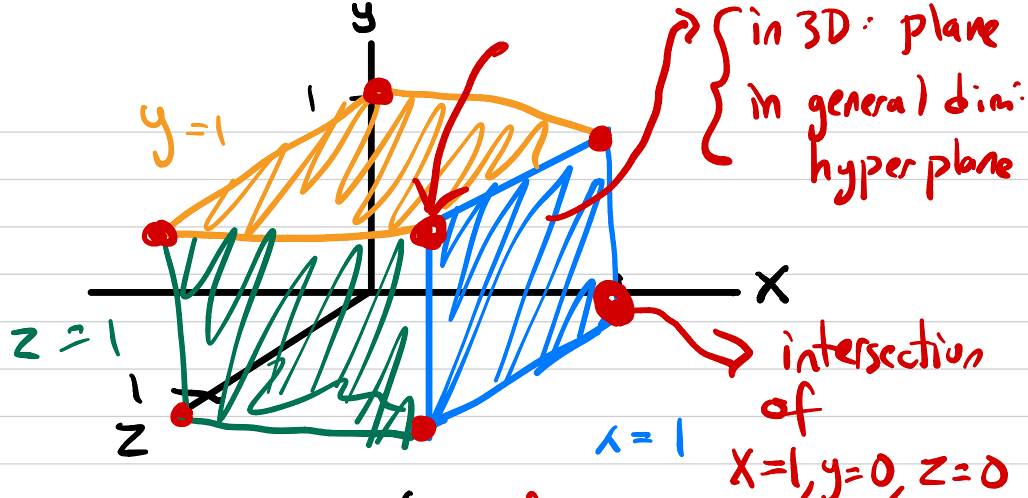
$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m
\end{aligned}\right\} m$$

Objective

Maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

LP example

$$\begin{aligned} \max \quad & x + y + z \\ \text{s.t.} \quad & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ & 0 \le z \le 1 \end{aligned}$$



Def: A vertex is a point x in the polytope ($x \in$ feasible set) which lies at the intersection of n hyperplanes in n dimensions

Recall: Always an optimal solution at a vertex

Fact: Suppose LP has n vars and m constraints.

Then # of vertices $\le \underbrace{\binom{m}{n}}_{\approx\, m^n} \approx$ exponential in n