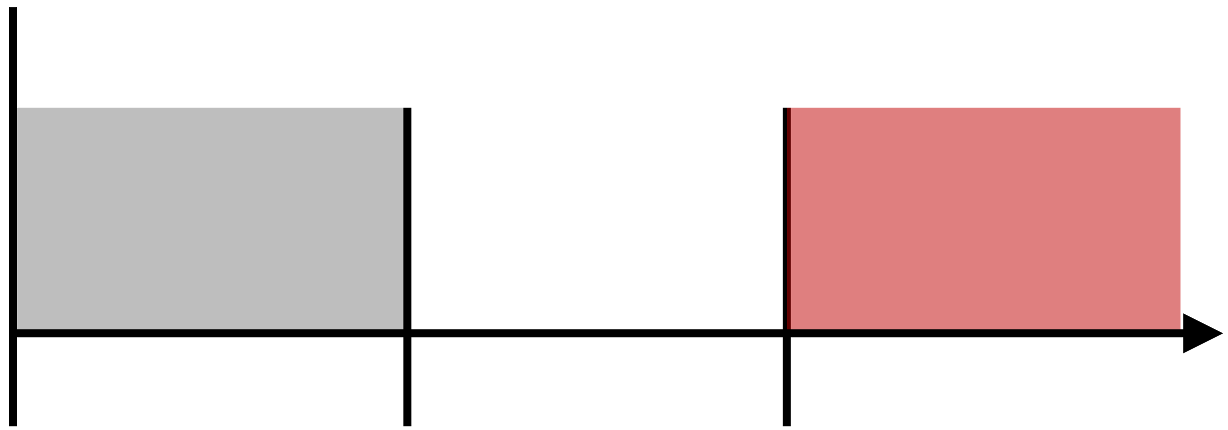
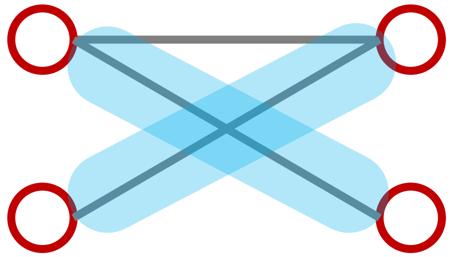
Lecture 17

Duality

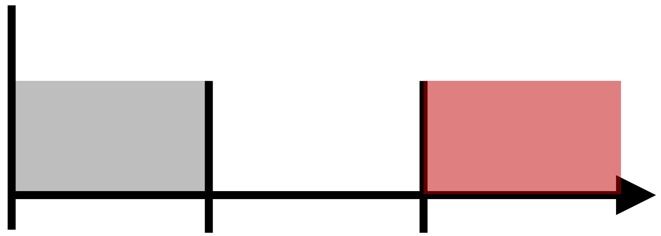


Outline

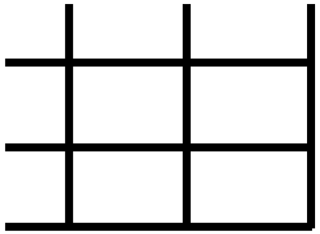
1. Bipartite perfect matching



2. Linear programming duality



3. Zero-sum games

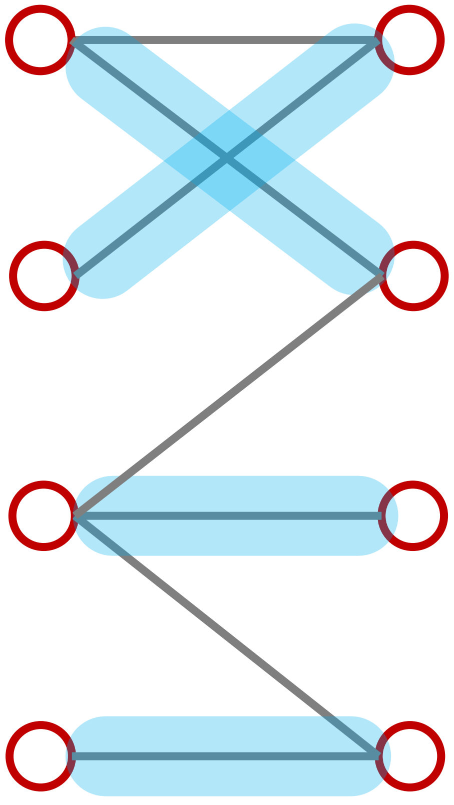


Bipartite Perfect Matching

Input: Bipartite (undirected) graph $G = (L, R, E)$ with $|L| = |R| = n$

Output: A perfect matching from L to R

Graph G



L R

Example:

L = UC Berkeley courses

R = UC Berkeley classrooms

E = each course is connected to the classrooms it can be taught in

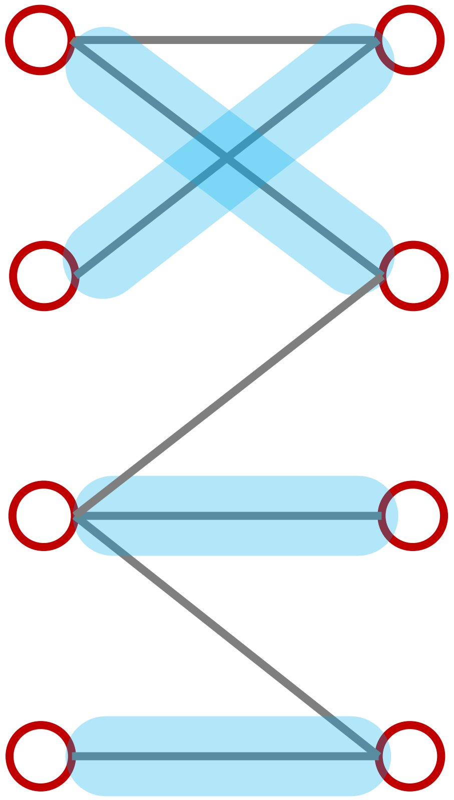
Q: Can we assign every course to a room?

Bipartite Perfect Matching

Input: Bipartite (undirected) graph $G = (L, R, E)$ with $|L| = |R| = n$

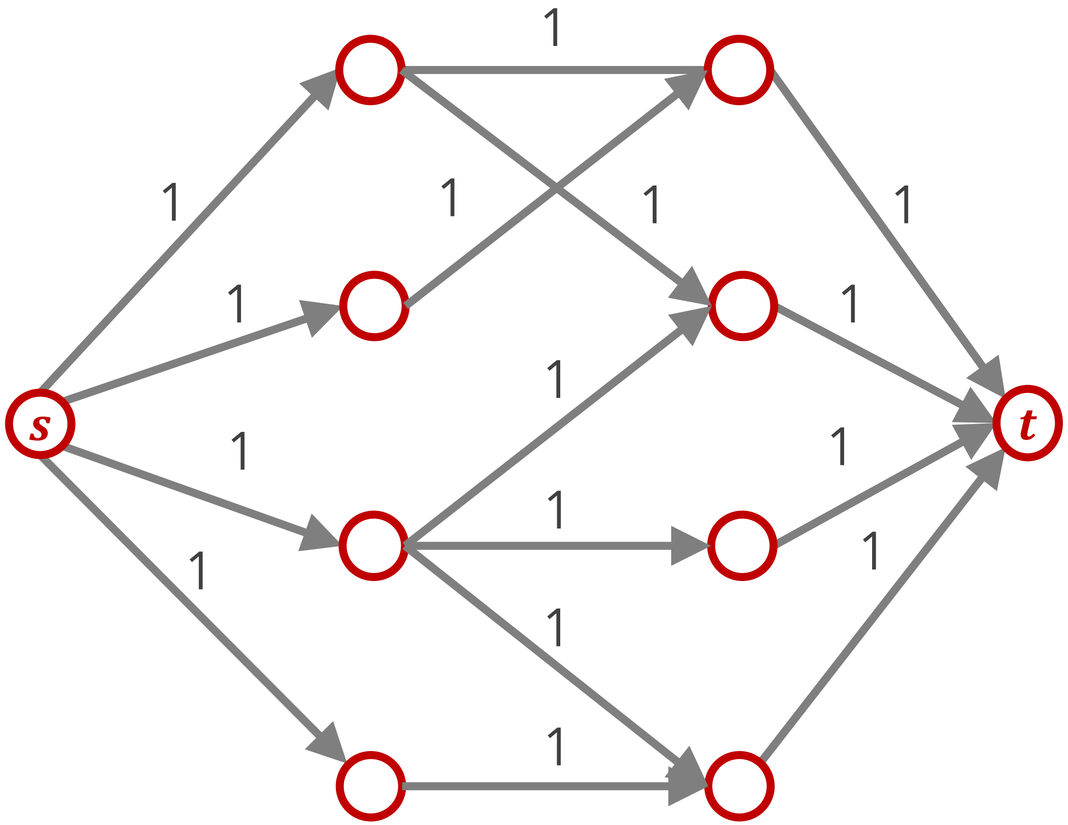
Output: A perfect matching from L to R

Graph
G



L R

Graph
G'



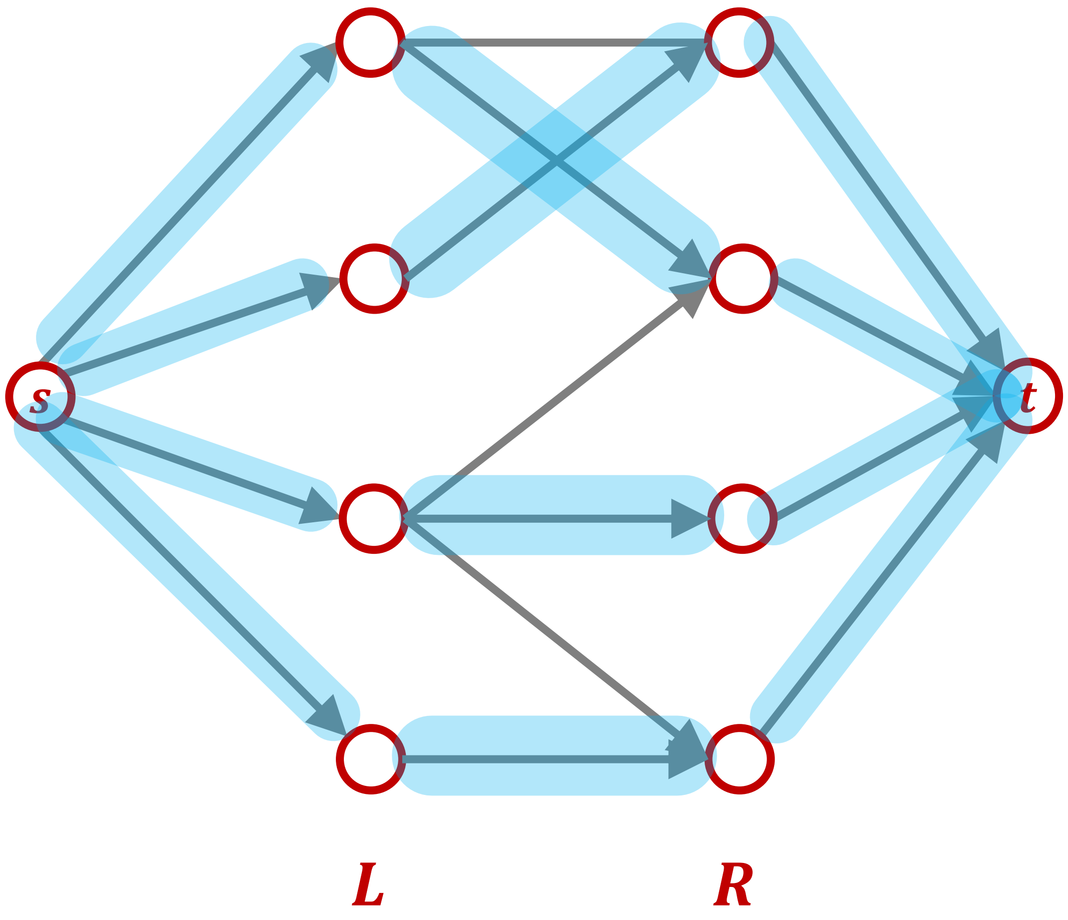
Thm: G has a perfect matching $\Leftrightarrow$ Max-flow$(G') = n$

Thm: G has a perfect matching
$\Leftrightarrow$ Max-flow$(G') = n$

Pf:

Case 1: $(\Rightarrow)$ ☑

1. Let M be a perfect matching in G.
2. Put **1** unit of flow on every edge in **M**
   and every $s \to v$ edge
   and every $v \to t$ edge.
3. Then this is a flow of size n.



Graph G'

Thm: G has a perfect matching
$\Leftrightarrow$ Max-flow$(G') = n$

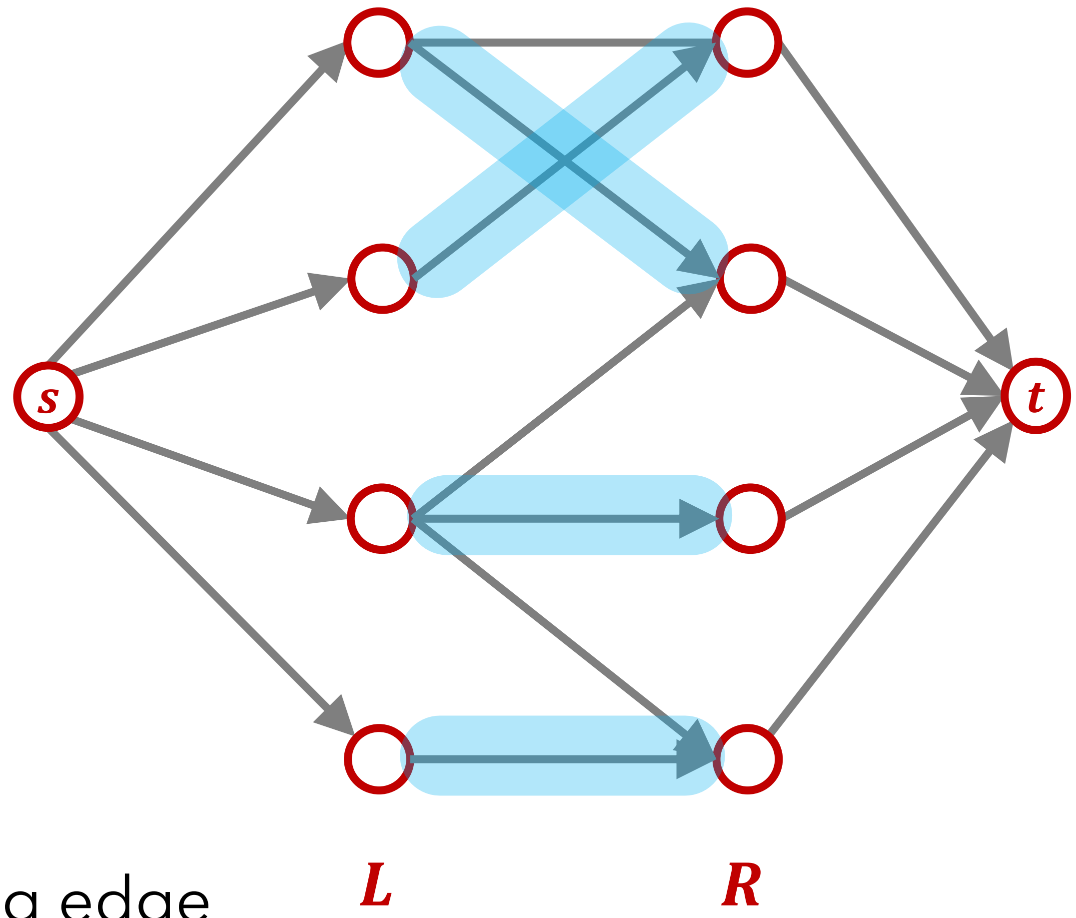
a "**reduction** from perfect matching to maximum flow"

Pf:

Case 1: ($\Rightarrow$) ☑

Case 2: ($\Leftarrow$) ☑

> **Recall from last lecture:**
>
> If the capacities are integral, then the Max-Flow is integral.

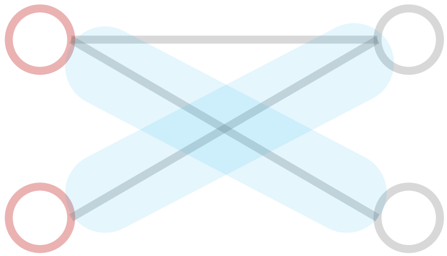


Graph G'

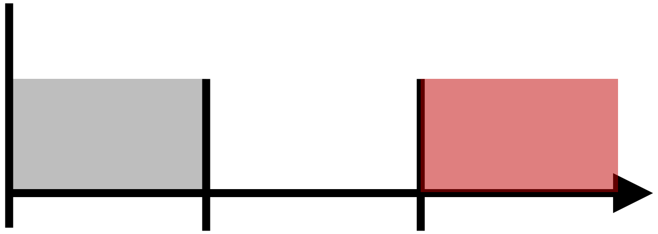
1. Let f be an **integral** flow of size n in G'.
   (all flow values 0 or 1)
2. Each $u \in L$ has **1** unit of flow on **1** outgoing edge
3. Each $v \in R$ has **1** unit of flow on **1** incoming edge
4. These edges form a matching of size n. $\square$

Outline

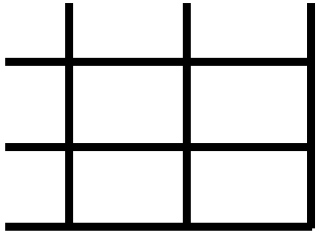
1. Bipartite perfect matching



2. Linear programming duality



3. Zero-sum games



Last class: Max-Flow = Min-Cut

Could always prove that a flow was **optimal** by showing a cut of the same value

This is a general property of LPs known as **duality**

The book calls duality a **magic trick**

$$
\begin{array}{llrcl}
\text{max} & 5x_1 + 4x_2 & & & \\
\text{s.t.} & 2x_1 + \quad x_2 & \leq & 100 & \qquad \text{also} \quad x_1 \geq 0 \\
& \quad x_1 & \leq & 30 & \qquad \qquad \quad\; x_2 \geq 0 \\
& \qquad\quad\;\; x_2 & \leq & 60 &
\end{array}
$$

Solution: $x_1 = 20, x_2 = 60, \text{value} = 340$

Q: Can we **prove** this is optimal?

$$
\begin{array}{lrl}
\max & 5x_1 + 4x_2 & \\
\text{s.t.} & 2x_1 + x_2 \le 100 & \\
 & (\ x_1 \le 30\ ) \cdot 5 & \\
+ & (\ x_2 \le 60\ ) \cdot 4 & \\
\hline
 & 5x_1 + 4x_2 \le \underbrace{5 \cdot 30 + 4 \cdot 60}_{390} &
\end{array}
\qquad
\text{also} \quad
\begin{array}{l}
x_1 \ge 0 \\
x_2 \ge 0
\end{array}
$$

Solution: $x_1 = 20, x_2 = 60$, value $= 340$

Q: Can we **prove** this is optimal?

$$
\begin{array}{lll}
\text{max} & 5x_1 + 4x_2 & \\
\text{s.t.} & (2x_1 + x_2 \leq 100) \cdot 3 & \text{also} \quad x_1 \geq 0 \\
 & (x_1 \leq 30) \cdot 0 & \qquad\quad x_2 \geq 0 \\
+ & (x_2 \leq 60) \cdot 1 &
\end{array}
$$

$$5x_1 + 4x_2 \leq 6x_1 + 4x_2 \leq \underbrace{3 \cdot 100 + 60}_{360}$$

Solution: $x_1 = 20, x_2 = 60$, value $= 340$

Q: Can we **prove** this is optimal?

$$
\begin{array}{lll}
\max & 5x_1 + 4x_2 & \\
\text{s.t.} & (2x_1 + x_2 \le 100) \cdot 5/2 & \text{also} \quad x_1 \ge 0 \\
 & (x_1 \le 30) \cdot 0 & \phantom{\text{also}} \quad x_2 \ge 0 \\
+ & (x_2 \le 60) \cdot 3/2 & \\
\hline
\end{array}
$$

$$
5x_1 + 4x_2 \le \underbrace{\frac{5}{2} \cdot 100 + \frac{3}{2} \cdot 60}_{340}
$$

Solution: $x_1 = 20, x_2 = 60, \text{value} = 340$

Q: Can we **prove** this is optimal?

Primal LP:

$$\begin{array}{rlcl}
\max & 5x_1 + 4x_2 & & \\
\text{s.t.} & (2x_1 + x_2 \le 100) \cdot y_1 & \text{also} & x_1 \ge 0 \\
& (x_1 \le 30) \cdot y_2 & & x_2 \ge 0 \\
+ & (x_2 \le 60) \cdot y_3 & & \\
\hline
\end{array}$$

$$(2y_1 + y_2) \cdot x_1 + (y_1 + y_3) \cdot x_2 \le 100 \cdot y_1 + 30 \cdot y_2 + 60 \cdot y_3$$

Dual LP:

$$\begin{array}{rl}
\min & 100 \cdot y_1 + 30 \cdot y_2 + 60 \cdot y_3 \\
\text{s.t.} & y_1, y_2, y_3 \ge 0 \\
& 5 \le 2y_1 + y_2 \\
& 4 \le y_1 + y_3
\end{array}$$

By construction: $5x_1 + 4x_2 \le 100 \cdot y_1 + 30 \cdot y_2 + 60 \cdot y_3$

Primal LP Opt $\le$ **Dual LP Opt**

Primal LP

$$\max \quad \begin{bmatrix} \; c^T \; \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} \; A \; \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} b \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} x \end{bmatrix} \geq 0$$

Dual LP

$$\max \quad \begin{bmatrix} \; b^T \; \end{bmatrix} \cdot \begin{bmatrix} y \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} \; A^T \; \end{bmatrix} \cdot \begin{bmatrix} y \end{bmatrix} \geq \begin{bmatrix} c \end{bmatrix}$$

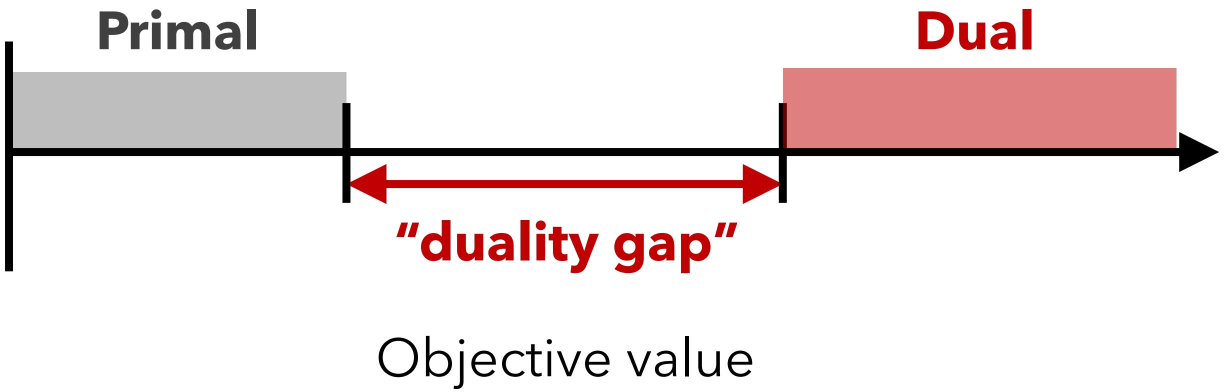
$$\text{and} \quad \begin{bmatrix} y \end{bmatrix} \geq 0$$

Thm: **(Weak duality)** all feasible solutions x to primal LP $\leq$ all feasible solutions y to dual LP

Pf: $$\begin{bmatrix} \; c^T \; \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} \; y^T \; \end{bmatrix} \begin{bmatrix} \; A \; \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} \; y^T \; \end{bmatrix} \begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} \; b^T \; \end{bmatrix} \begin{bmatrix} y \end{bmatrix} \qquad \square$$

Cor: **Primal LP OPT $\leq$ Dual LP OPT**

Weak duality:



Thm: **(Strong duality)**

If the **Primal LP Opt** is bounded,
then **Primal LP Opt** = **Dual LP Opt**

∴ **duality gap** = 0

Example: Max-Flow **=** Min-Cut (in recitation)

LP $\xleftrightarrow{\textbf{dual}}$ **LP**

LP duality history

George Dantzig

Co-inventor of LPs,
inventor of simplex,
Berkeley grad student
and faculty

Was taking a statistics class

Professor wrote two of the most famous unsolved problems in statistics on the board

But Dantzig arrived late, mistook them for homework

Turned in solutions a few days later, said they "seemed to be a little harder than usual"

LP duality history

George Dantzig

Co-inventor of LPs, inventor of simplex, Berkeley grad student and faculty

"Let me tell you about my newest invention: linear programming"

"Oh that!"

(Lectures Dantzig about linear programming for 1.5 hours, invents linear program duality)

"It's equivalent to zero-sum games, which I have also recently invented"

John von Neumann

All-time great mathematician

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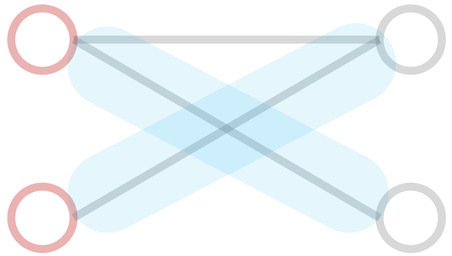

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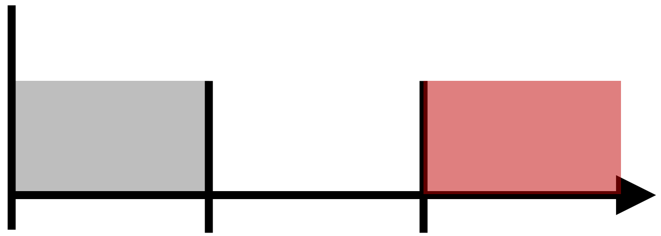

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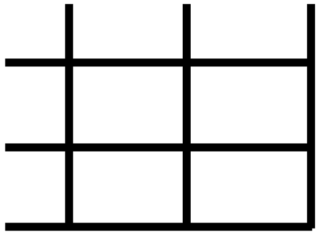

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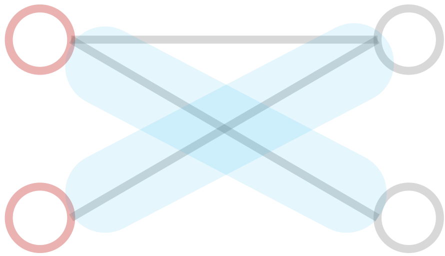

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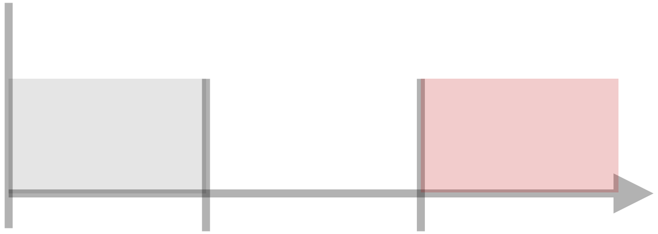

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