Lecture 16
Maximum flow

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U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

FUNDAMENTALS OF A METHOD FOR EVALUATING RAIL NET CAPACITIES (U)

T. E. Harris
F. S. Ross

RM-1573

October 24, 1955

Copy No. 37

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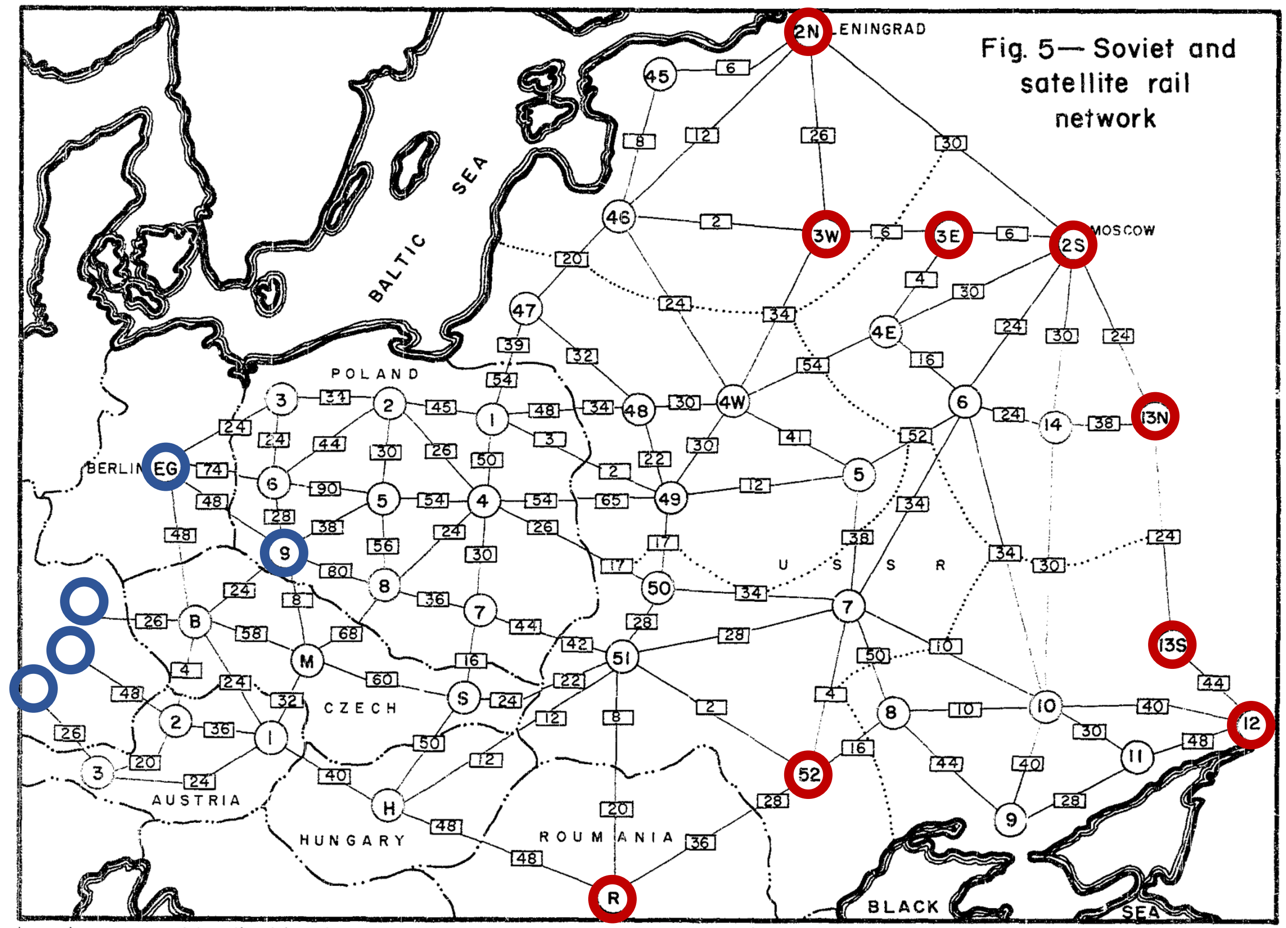

Legend: —·—·— International boundary ········· Regional boundaries of the USSR (they are included as a matter of general information)

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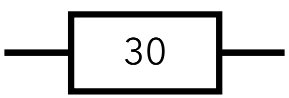

= a capacity of 30,000 tons

: source

: destination

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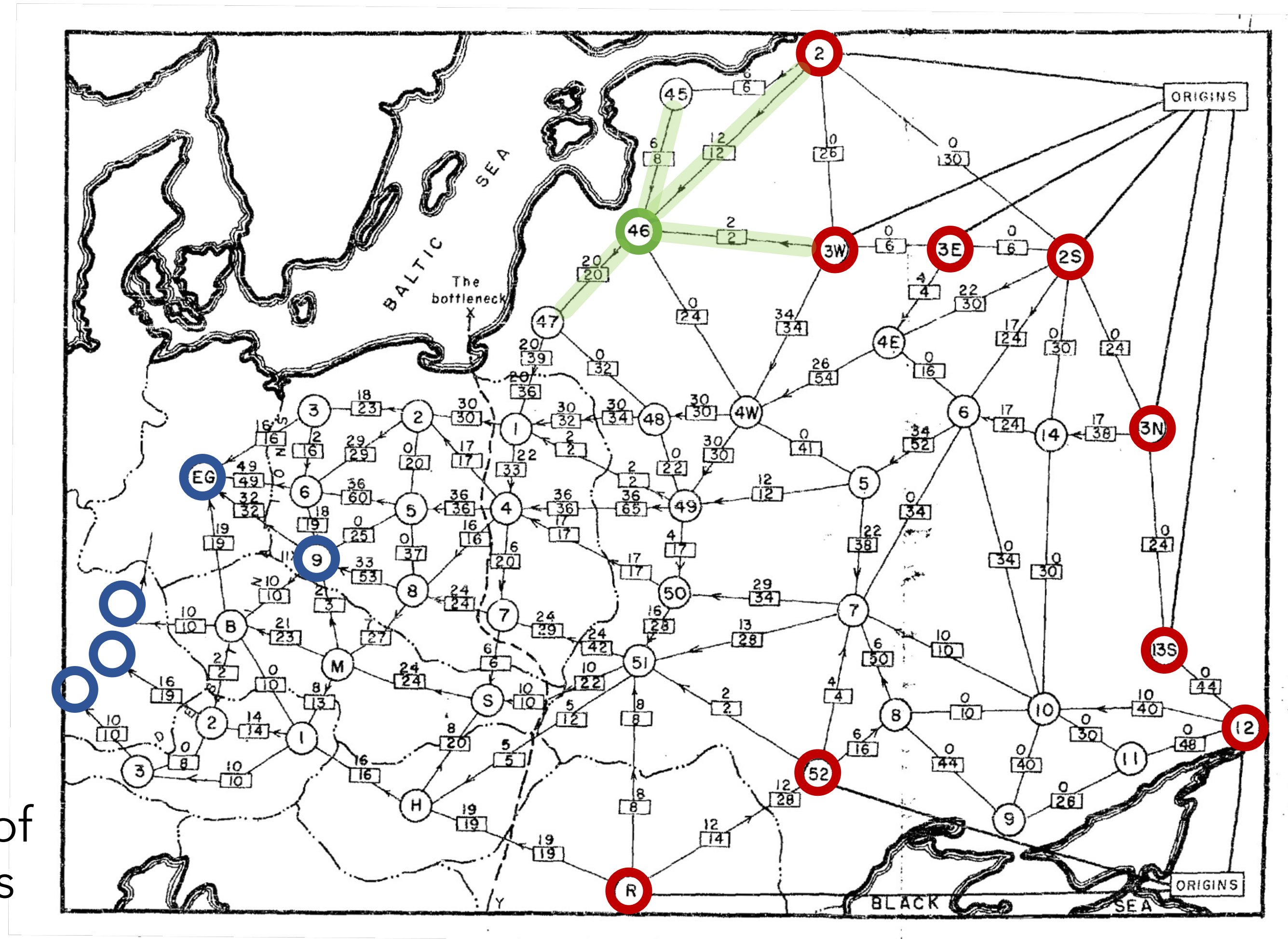

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30

= a capacity of 30,000 tons

: source

: destination

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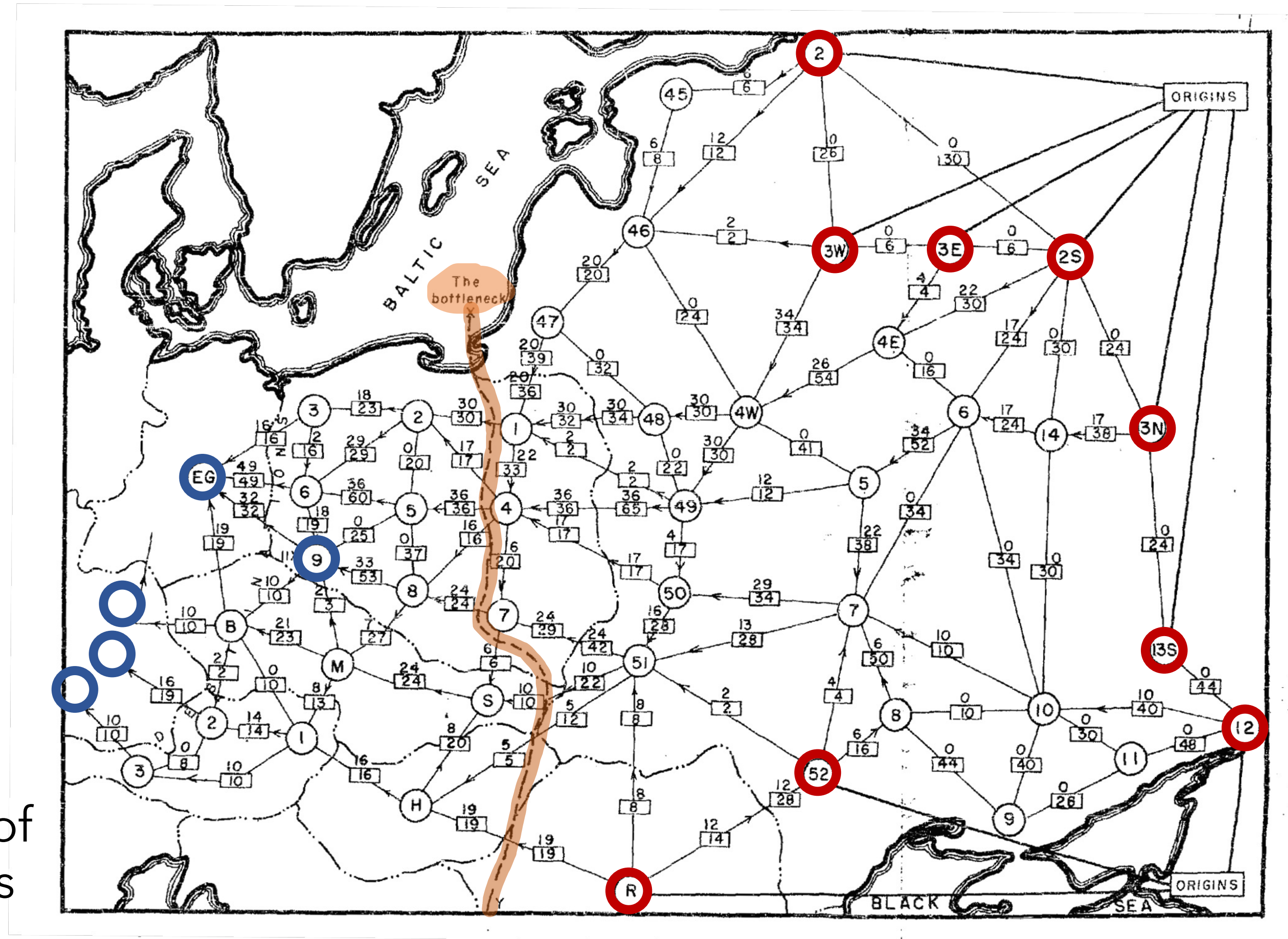

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30
= a capacity of 30,000 tons

: source
: destination

Total East → West capacity: **163,000 tons**

Optimal due to **the bottleneck**

Harris and Ross solved this problem
using a **greedy** algorithm they called "flooding"

But flooding would sometimes output **incorrect** solutions

So they approached their colleagues Ford and Fulkerson,
who devised an alg known as the **Ford-Fulkerson algorithm**

MAXIMAL FLOW THROUGH A NETWORK

L. R. FORD, JR. AND D. R. FULKERSON

Introduction. The problem discussed in this paper was formulated by T. Harris as follows:

We will see this algorithm today.

See also: "On the history of the transportation and maximum flow problems"
by Alexander Schrijver

Maximum flow

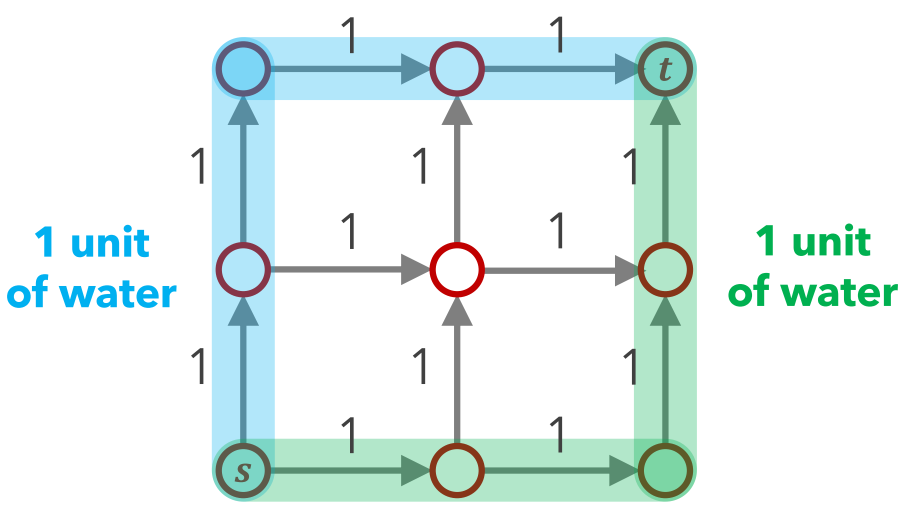
Input: 1. Directed graph $G = (V, E)$

2. One "source vertex" $s \in V$

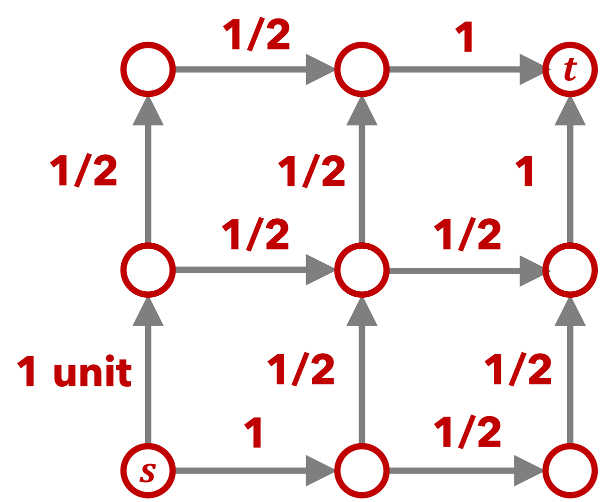
3. One "sink vertex" $t \in V$

4. For each edge $e \in E$, a "capacity" $c_e \in \mathbb{Z}^+$ (or $\mathbb{R}^+$)

Goal: Route the maximum amount of water from s to t



2 units of flow (water)



Maximum flow

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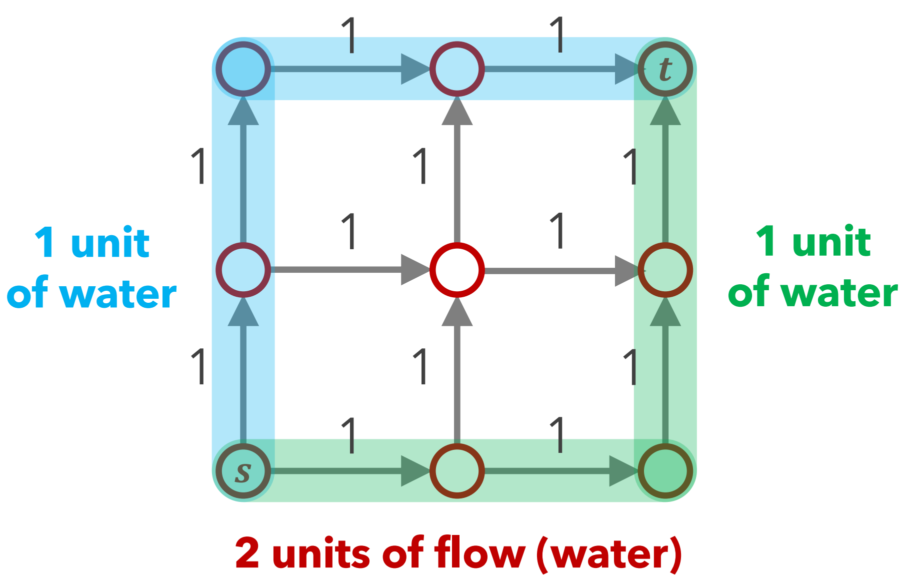
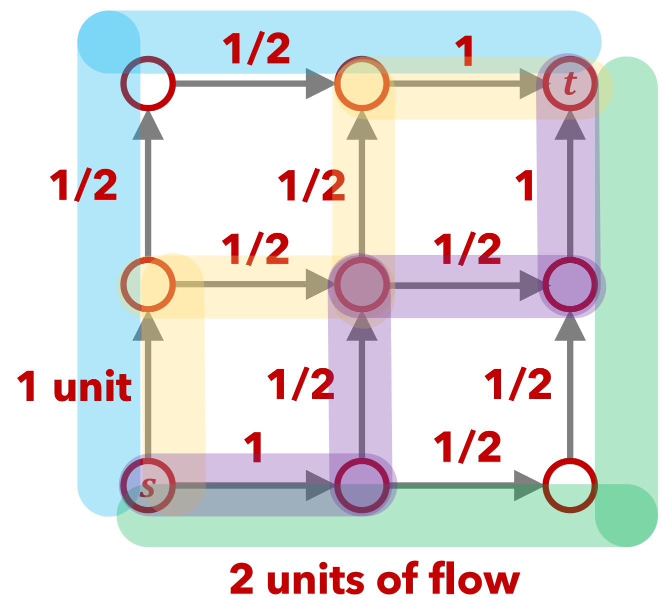


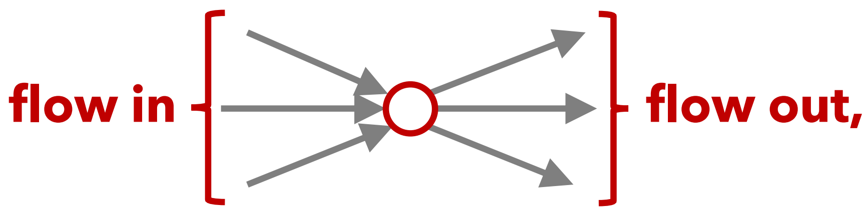


Def: A **flow** assigns a number f_e to each directed edge $e \in E$ such that

(Nonnegativity) $f_e \geq 0$

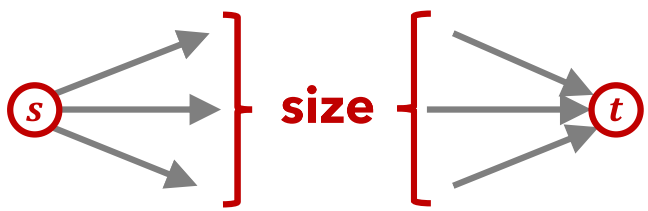
(Capacity) $f_e \leq c_e$

(Flow in = flow out) for each vertex $v \neq s, t$,



$$\sum_{u \to v} f_{u,v} = \sum_{v \to w} f_{v,w}$$

Def: The **size** of a flow f is the total quantity sent from s to t.



$$\mathbf{size}(f) = \sum_{s \to v} f_{s,v} = \sum_{v \to t} f_{v,t}$$

Maximum flow

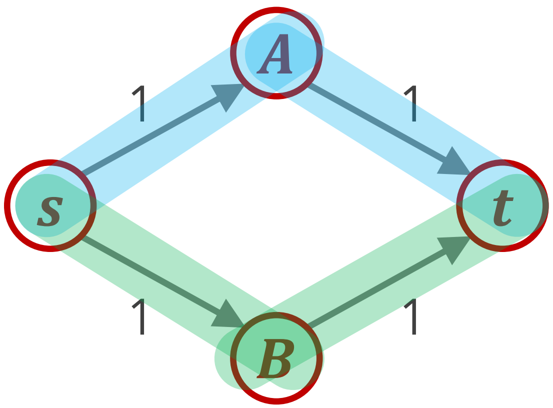
maximize $\mathbf{size}(f)$

s.t. $\{f_e\}$ is a flow

= a linear program!

Max flow algorithm: first try (Harris and Ross' "flooding" method)

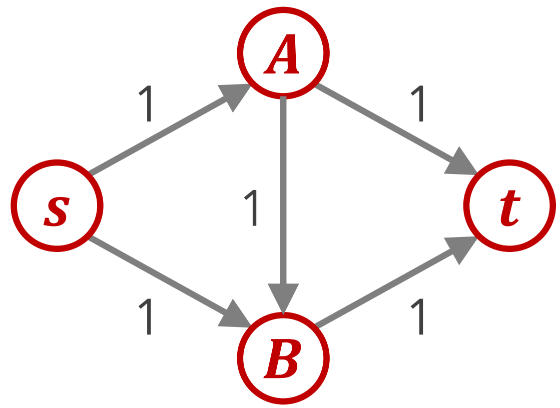
1. Find a path ***P*** from ***s*** to ***t*** which is not yet saturated
2. Send more flow along ***P***
3. Repeat

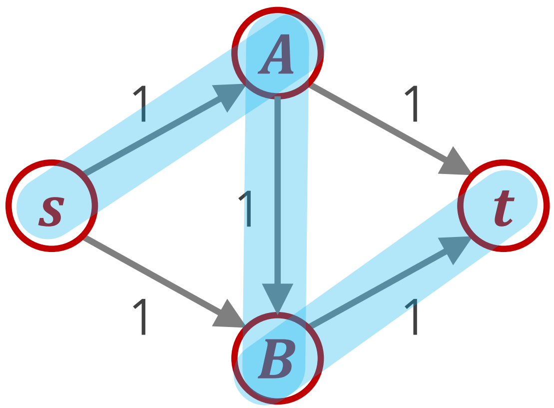


$s \to A \to t$: **1 unit**

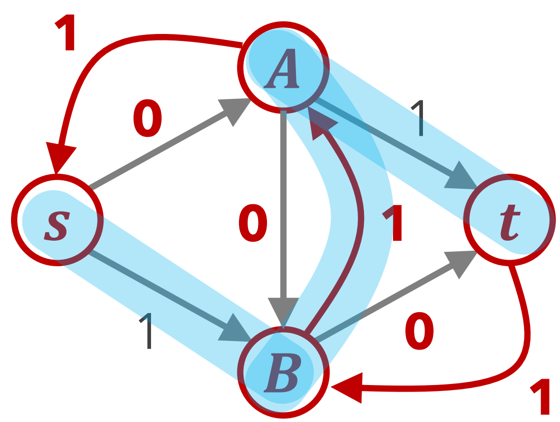
$s \to B \to t$: **1 unit**

total: **2 units**



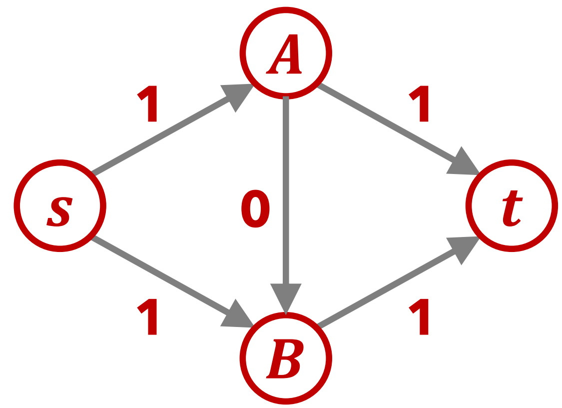


$s \rightarrow A \rightarrow B \rightarrow t$: **1 unit**



"residual graph"

$s \rightarrow B \rightarrow A \rightarrow t$: **1 unit**

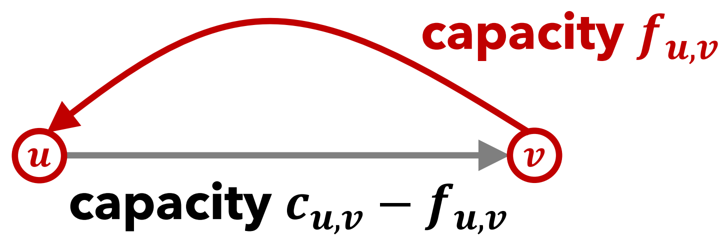


2 units total!

Def: Given a graph $\boldsymbol{G}$ and a flow $\boldsymbol{f}$ on $\boldsymbol{G}$, the residual graph $\boldsymbol{G_f}$ is as follows.

For all edges $(\boldsymbol{u}, \boldsymbol{v})$:

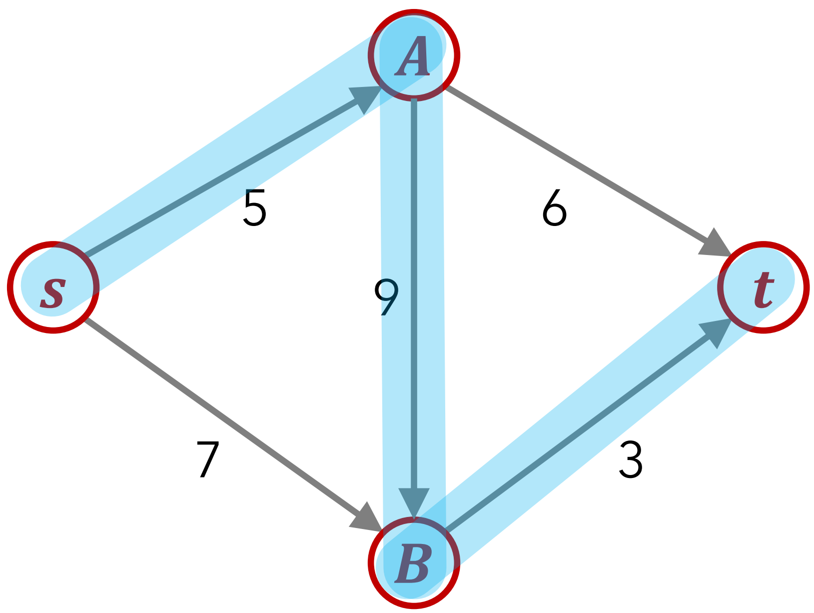
(in original graph)



(in residual graph)

Ford-Fulkerson algorithm

1. Find a path P from s to t **in the residual graph** which is not yet saturated
2. Send more flow along P = an **augmenting path**
3. Repeat



$s \to A \to B \to t$: **3 units**

Ford-Fulkerson algorithm

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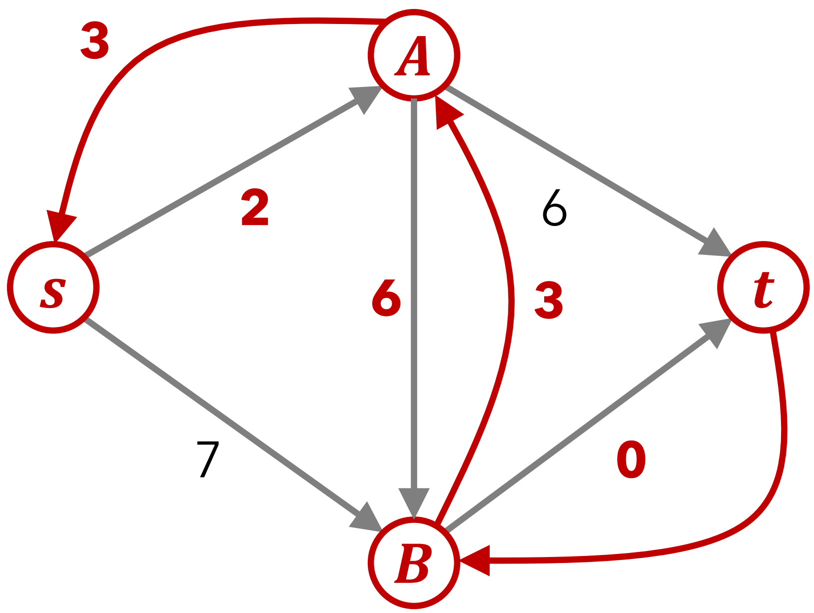

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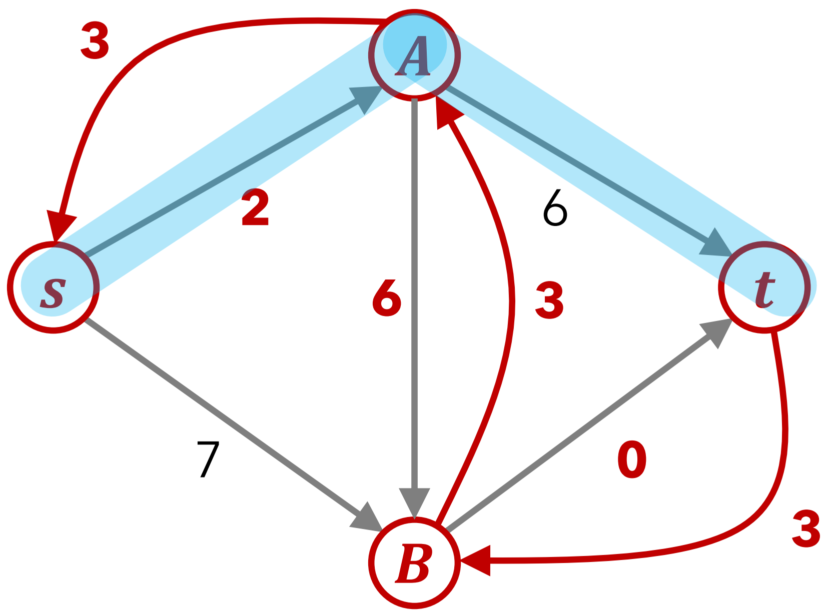

$s \to A \to B \to t$: **3 units**

$s \to A \to t$: **2 units**

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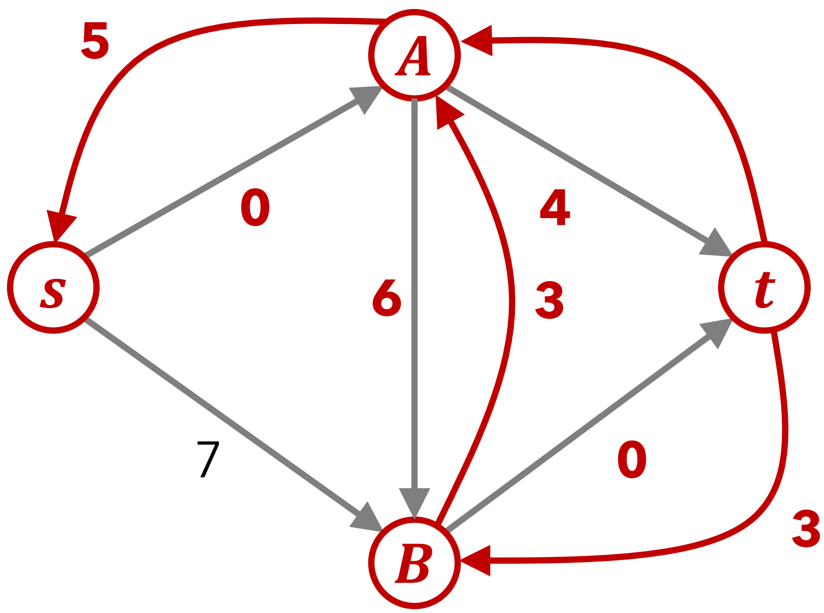

$s \to A \to B \to t$: **3 units**

$s \to A \to t$: **2 units**

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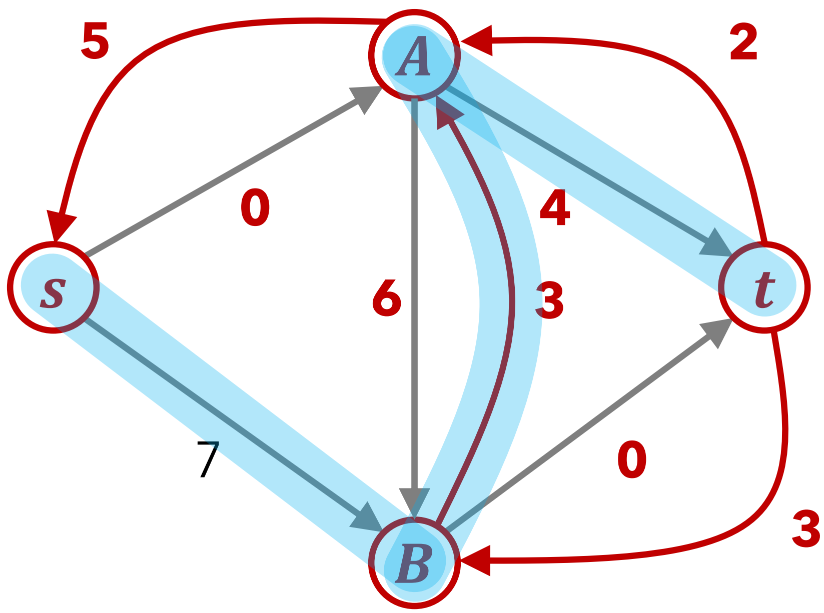

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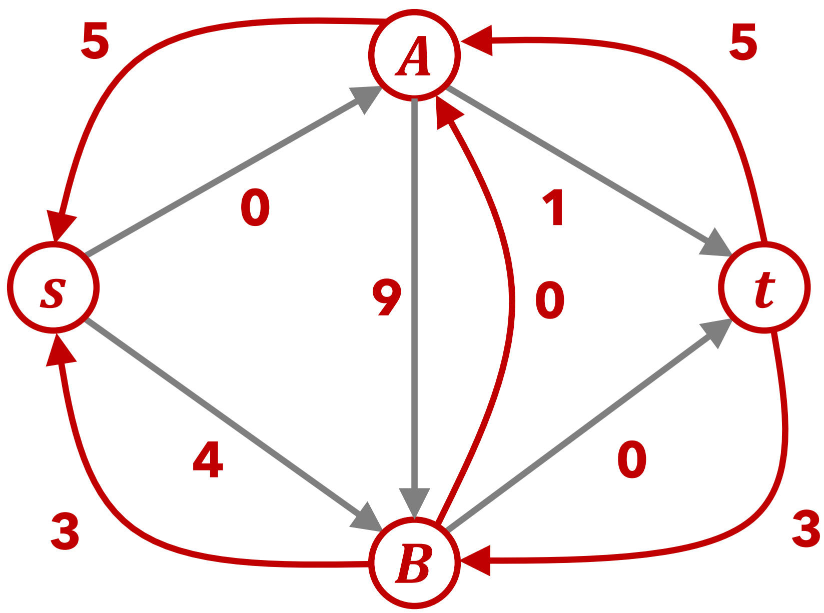

$s \to A \to B \to t$: **3 units**

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total: **8 units**

Ford-Fulkerson algorithm

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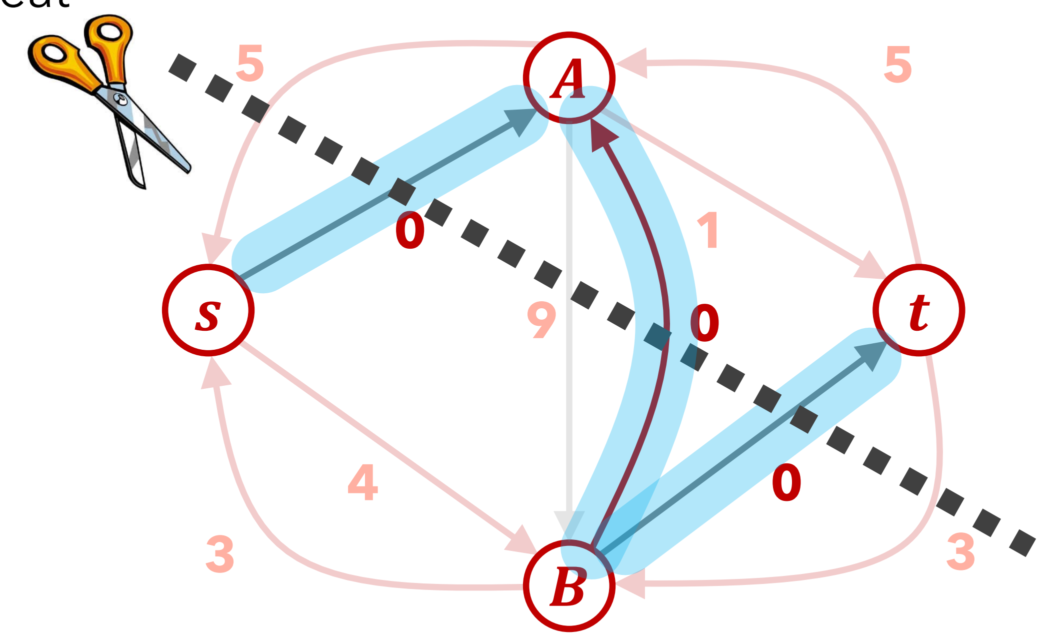


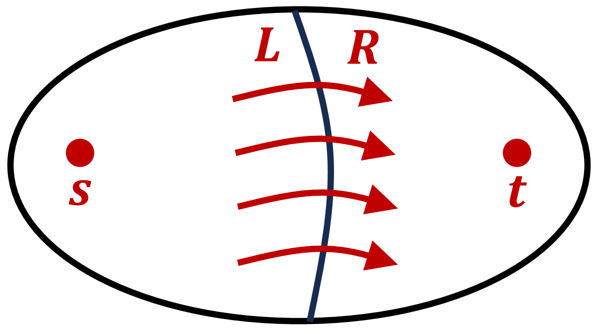
$s \to A \to B \to t$: **3 units**

$s \to A \to t$: **2 units**

$s \to B \to A \to t$: **3 units**

total: **8 units**

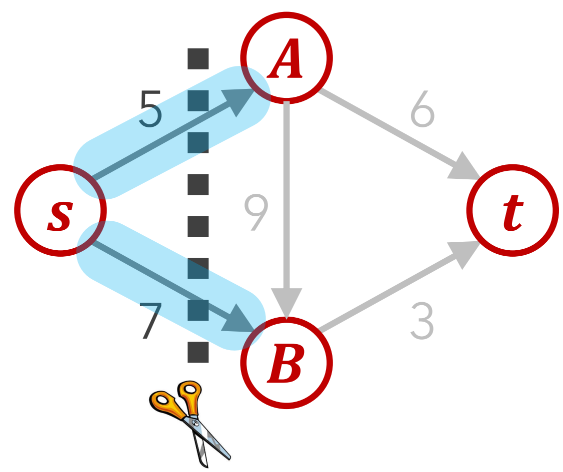
Def: An ***s*-*t* cut** is a partition $V = L \cup R$ of the vertices such that $s \in L$ and $t \in R$



Def: The **capacity** of the **cut** is $\text{capacity}(L, R) = \sum_{\substack{u \to v \\ u \in L, v \in R}} c_{u,v}$

Thm: For any flow f and any cut (L, R),

$$\text{size}(f) \le \text{capacity}(L, R).$$



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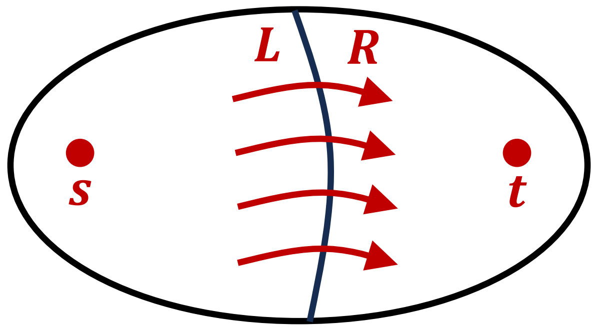


Def: The **capacity** of the **cut** is $\textbf{capacity}(L,R) = \sum_{\substack{u \to v \\ u \in L, v \in R}} c_{u,v}$

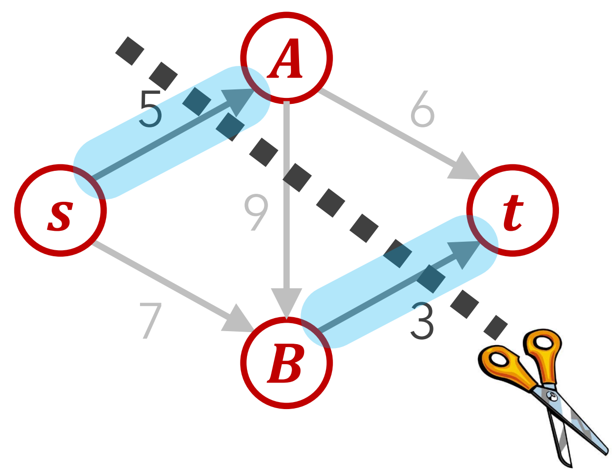
Thm: For any flow f and any cut (L,R),

$$\textbf{size}(f) \leq \textbf{capacity}(L,R).$$

Def: The **Min-cut** is the cut with **minimum** capacity.

Aka: Max-flow ≤ Min-cut

(Harris and Ross' "bottleneck")



Thm: Max-flow = Min-cut

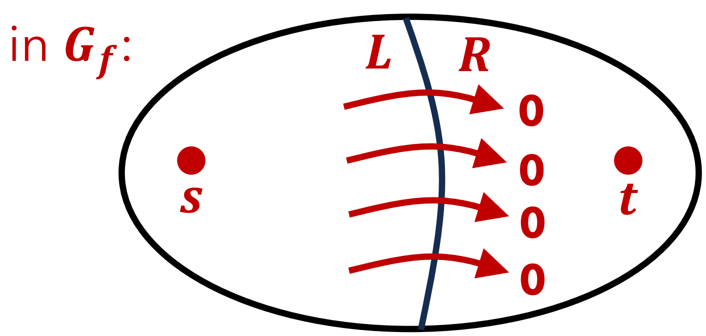
Pf: Only need to show "≥"

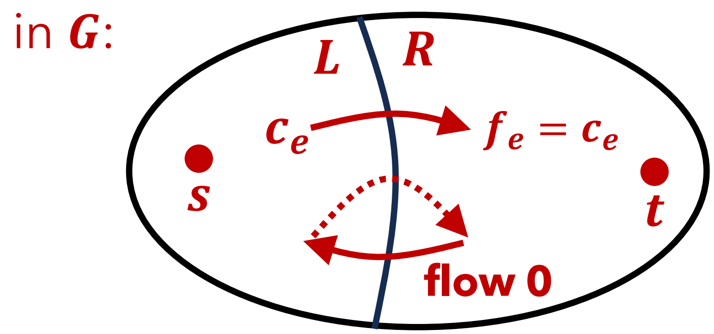
Run Ford-Fulkerson on G. Let f be the flow it outputs.

Then no $s \to t$ in residual graph G_f.

Set L = vertices reachable from s in G_f.

R = everything else.





Max-flow ≥ $\text{size}(f) = \text{capacity}(L, R)$ ≥ Min-cut. □

Thm: Ford-Fulkerson outputs a **maximum flow**.

Runtime $\approx$ # of augmenting paths $\leq \boldsymbol{U}$, where $\boldsymbol{U}$ = Max-flow

(× the time to find the paths) (in graphs of integer weights)

$\leq \mathbf{O}(m+n) \cdot \boldsymbol{U}$

Is this a good runtime?

Suppose each capacity $\boldsymbol{c_e}$ was $\leq \boldsymbol{C}$.

Then $\boldsymbol{U} \leq \boldsymbol{m} \cdot \boldsymbol{C}$.

Each $\boldsymbol{c_e}$ is a $\mathbf{log_2}(\boldsymbol{C})$–bit integer.

So $\boldsymbol{U}$ can be **exponential** in the input length!

This is a **pseudo-polynomial** algorithm

(it is polynomial in the **numerical value** of the input)

Recall: Knapsack

Runtime $\approx$ # of augmenting paths $\leq \boldsymbol{U}$, where $\boldsymbol{U}$ = Max-flow

(× the time to find the paths) (in graphs of integer weights)

$\leq \mathbf{O}(m+n) \cdot \boldsymbol{U}$

Surprise: If all the capacities are integral, then the Max-flow is integral.

(all the capacities/flows are integers)

Other algorithms:

Dinitz 1970/Edmonds-Karp 1972: Always pick the shortest augmenting path

Runs in time $\mathbf{O}(n\, m^2)$!

⋮

(many, many more)

⋮

Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva 2022: $\mathbf{O}(m^{1+o(1)} \cdot \log(\boldsymbol{U}))$

(only 112 pages!)