Two-player

Zero-Sum

Games

Zero-sum games

Input: A payoff matrix M

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$$\left.\begin{array}{l}\text{Row player: picks row } r\\ \text{Col player: picks col } c\end{array}\right\} \rightarrow \left\{\begin{array}{l}\text{Win } +A[r,c]\\ \text{Win } -A[r,c]\end{array}\right.$$

Types of strategies:

"Pure strategy": a single row/column

e.g. row player always picks rock (beaten by paper col)

"Mixed strategy": probability distribution over pure strategies

e.g. $\Pr[\text{Rock}] = 1/3$, $\Pr[\text{Paper}] = 1/3$, $\Pr[\text{Scissors}] = 1/3$

Note: average score $= 0$ (no matter what col player does)

Game 1

Turn Order:

1. Row player announces mixed strat $p = (p_1, p_2)$
2. Col player responds w/ mixed strat $q = (q_1, q_2)$

$p_1 = \Pr[\text{Row } 1]$

$p_2 = \Pr[\text{Row } 2]$

	1	2
1	3	−1
2	−2	1
	q_1	q_2

Def: Row player's average score is $\text{Score}(p,q) = 3 \cdot p_1 q_1 - 1 \cdot p_1 q_2 - 2 \cdot p_2 q_1 + 1 \cdot p_1 q_2$

Col player's best strat: $\underset{\text{mixed strat } q}{\text{minimize}} \left\{ \text{Score}(p,q) \right\}$

(should pick best col) $= \underset{\text{pure strat } q}{\min} \left\{ \text{Score}(p,q) \right\} = \min \left\{ \underbrace{3 \cdot p_1 - 2 \cdot p_2}_{\text{col 1 score}}, \underbrace{-1 \cdot p_1 + 1 \cdot p_2}_{\text{col 2 score}} \right\}$

Row player's best strat: $\underset{\text{mixed strat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1, -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$

Fact: Can calculate $\underset{\text{mixed strat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1, -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$ with LP.

Pf:

$$
\begin{aligned}
\text{maximize} \quad & z \\
\text{subject to} \quad & z \le 3 p_1 - 2 p_2 \\
& z \le - p_1 + p_2 \\
& p_1 + p_2 = 1 \\
& p_1 \geq 0, \; p_2 \geq 0 .
\end{aligned}
\qquad = LP_1
$$

Note: $z = \min \{ 3 p_1 - 2 p_2, -p_1 + p_2 \}$.

$\square$

Game 2

Same as Game 1, except col player goes 1st and row player goes 2nd

$$\begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & 3 & -1 & p_1 \\ 2 & -2 & 1 & p_2 \\ \hline & q_1 & q_2 & \end{array}$$

Payoff of row 1: $3 \cdot q_1 - 1 \cdot q_2$

Payoff of row 2: $-2 \cdot q_1 + 1 \cdot q_2$

Row player's best strat = $\max \{ 3q_1 - q_2, -2q_1 + q_2 \}$

Col " " " = $\underset{\text{mixed strat } q}{\text{minimize}} \{ \max \{ 3q_1 - q_2, -2q_1 + q_2 \} \}$

LP_2 = minimize Z

subject to
$$3q_1 - q_2 \leq Z$$
$$-2q_1 + q_2 \leq Z$$
$$q_1 + q_2 = 1, \quad q_1 \geq 0, q_2 \geq 0.$$

Game 1

1. Row player first
2. Col player second

	1	2
1	3	-1
2	-2	1

Game 2

1. Col player first
2. Row player second

$$\max_{p}\left\{\min_{q}\left\{Score(p,q)\right\}\right\} \le \min_{q}\left\{\max_{p}\left\{Score(p,q)\right\}\right\}$$

$$LP_1 \xleftrightarrow{\text{(dual)}} LP_2$$

$\therefore$ Strong duality $\Rightarrow$ $LP_1 = LP_2 = Value(Game)$ (Definition)

(Min-Max Theorem)

$\Rightarrow$ order of play doesn't change value

$\Rightarrow$ $\exists$ optimal strat for ROW, irrespective of COL

Minimax strats

$$\begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & 3 & -1 & p_1 \\ 2 & -2 & 1 & p_2 \\ \hline & q_1 & q_2 & \end{array}$$

Row player's optimum strategy

$p_1 = 3/7 \qquad p_2 = \frac{4}{7}$

Payoff of col 1 $= \frac{3}{7} \cdot 3 + \frac{4}{7} \cdot (-2) = \frac{1}{7}$

Payoff of col 2 $= \frac{3}{7} \cdot (-1) + \frac{4}{7} \cdot 1 = \frac{1}{7}$ (doesn't matter what col player does!)

Col player's optimum strategy

$q_1 = \frac{2}{7} \qquad q_2 = \frac{5}{7}$

Value(Game) $= \frac{1}{7}$

Payoff of row 1 $= \frac{2}{7} \cdot 3 + \frac{5}{7} \cdot (-1) = \frac{1}{7}$

Payoff of row 2 $= \frac{2}{7} \cdot (-2) + \frac{5}{7} \cdot (1) = \frac{1}{7}$ (doesn't matter what row player does!)

(Similar to RPS strategy w/ $\Pr[\text{Rock}] = \Pr[\text{Paper}] = \Pr[\text{Scissors}] = 1/3$)

Example problem

Suppose row player goes 1st and plays

$P_1 = \frac{1}{2}$ $P_2 = \frac{1}{5}$ $P_3 = \frac{3}{10}$

P_1	X	10	20
P_2	40	30	10
P_3	20	40	10

If the col player's optimal response is not unique, what is X?

$14 + \frac{X}{2} = 15$

$\therefore X = 2$

Payoff of col 1 $= \frac{X}{2} + 8 + 6 = 14 + \frac{X}{2}$

Payoff of col 2 $= 5 + 6 + 12 = 23$ (never use!)

Payoff of col 3 $= 10 + 2 + 3 = 15$

Extra stuff

Def: Nash equilibrium = pair of mixed strats (p, q) s.t. if row player plays p, col player plays q, neither player has incentive to deviate

In 2PZSG, Nash equilibria = minimax strats

Non zero-sum game

"Prisoner's dilemma"

	cooperate	defect
cooperate	$(-1, -1)$	$(-10, 0)$
defect	$(0, -10)$	$(-5, -5)$

col payoff

(a, b)

row's payoff

Nash equilibria = both players defect