

Two-player

Zero-Sum

Games

Zero-sum games

Input: A payoff matrix M

Row player: picks row r
Col player: picks col c } $\rightarrow \begin{cases} \text{Win} + A[r,c] \\ \text{Win} - A[r,c] \end{cases}$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Types of strategies:

"Pure strategy": a single row/column

e.g. row player always picks rock (beaten by paper col)

"Mixed strategy": probability distribution over pure strategies

e.g. $\Pr[\text{Rock}] = \frac{1}{3}$, $\Pr[\text{Paper}] = \frac{1}{3}$, $\Pr[\text{Scissors}] = \frac{1}{3}$

Note: average score = 0 (no matter what col player does)

Game 1

- Turn • 1. Row player announces
Order • mixed strat $P = (p_1, p_2)$
2. Col player responds
w/ mixed strat $q = (q_1, q_2)$

$$p_1 = Pr[\text{Row 1}]$$

$$p_2 = Pr[\text{Row 2}]$$

	1	2
1	3	-1
2	-2	1
	q_1	q_2

Def: Row player's average score is $\text{Score}(p, q) = 3 \cdot p_1 q_1 - 1 \cdot p_1 q_2 - 2 \cdot p_2 q_1 + 1 \cdot p_2 q_2$

Col player's best strat: minimize $\text{Score}(p, q)$

(should pick best col) $= \min_{\text{pure strat } q} \text{Score}(p, q) = \min \left\{ \underbrace{3 \cdot p_1 - 2 \cdot p_2}_{\text{col 1 score}}, \underbrace{-1 \cdot p_1 + 1 \cdot p_2}_{\text{col 2 score}} \right\}$

Row player's best strat: maximize $\left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1, -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$

Fact: Can calculate $\underset{\text{mixed stat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_2, -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$ with LP.

Pf: maximize z

subject to $z \leq 3 p_1 - 2 p_2$

$z \leq -p_1 + p_2$

$p_1 + p_2 = 1$

$p_1 \geq 0, p_2 \geq 0$.

= LP

Note: $z = \min \{ 3 p_1 - 2 p_2, -p_1 + p_2 \}$.

□

Game 2

Same as Game 1, except col player goes 1st and row player goes 2nd

		1	2	
1	3	-1	P	
2	-2	1		
		q_1	q_2	

Payoff of row 1: $3 \cdot q_1 - 1 \cdot q_2$

Payoff of row 2: $-2 \cdot q_1 + 1 \cdot q_2$

Row player's best stat = $\max \{ 3q_1 - q_2, -2q_1 + q_2 \}$

Col " " " = minimize $\left\{ \max \{ 3q_1 - q_2, -2q_1 + q_2 \} \right\}$
mixed strat q

$LP_2 =$ minimize Z

subject to $3q_1 - q_2 \leq Z$

$-2q_1 + q_2 \leq Z$

$q_1 + q_2 = 1, q_1 \geq 0, q_2 \geq 0.$

Game 1

1. Row player first
2. Col player second

	1	2
1	3	-1
2	-2	1

Game 2

1. Col player first
2. Row player second

$$\max_P \left\{ \min_Q \{ \text{Score}(p, q) \} \right\} \leq \min_Q \left\{ \max_P \{ \text{Score}(p, q) \} \right\}$$

// $LP_1 \xleftrightarrow{\text{(dual)}} LP_2$ //

$\therefore$ Strong duality $\Rightarrow LP_1 = LP_2 = \text{Value}(\text{Game})$ (Definition)
(Min-Max Theorem)

$\Rightarrow$ order of play doesn't change value

$\Rightarrow \exists$ optimal strat for ROW, irrespective of COL

Minimax strats

Row player's optimum strategy

$$p_1 = \frac{3}{7} \quad p_2 = \frac{4}{7}$$

$$\text{Payoff of col 1} = \frac{3}{7} \cdot 3 + \frac{4}{7} \cdot (-2) = \frac{1}{7}$$

$$\text{Payoff of col 2} = \frac{3}{7} \cdot (-1) + \frac{4}{7} \cdot 1 = \frac{1}{7}$$

		1	2	
1 2	1	3	-1	p_1
	2	-2	1	p_2
		q_1	q_2	

(doesn't matter what col player does!)

Col player's optimum strategy

$$q_1 = \frac{2}{7} \quad q_2 = \frac{5}{7}$$

$$\text{Payoff of row 1} = \frac{2}{7} \cdot 3 + \frac{5}{7} \cdot (-1) = \frac{1}{7}$$

$$\text{Payoff of row 2} = \frac{2}{7} \cdot (-2) + \frac{5}{7} \cdot (1) = \frac{1}{7} \quad (\text{doesn't matter what row player does!})$$

(similar to RPS strategy w/ $\Pr[\text{Rock}] = \Pr[\text{Paper}] = \Pr[\text{Scissors}] = \frac{1}{3}$)

$$\text{Value}(\text{Game}) = \frac{1}{7}$$

Example problem

Suppose row player goes 1^{st} and plays

$$P_1 = \frac{1}{2} \quad P_2 = \frac{1}{5} \quad P_3 = \frac{3}{10}$$

If the col player's optimal response is not unique, what is X ?

P_1	X	10	20
P_2	40	30	10
P_3	20	40	10

Extra stuff

Def: Nash equilibrium = pair of mixed strats (p, q) s.t. if row player plays p , col player plays q , neither player has incentive to deviate

In 2P2SG, Nash equilibria = minimax strats

Non zero-sum game
"Prisoner's dilemma"

Nash equilibria
= both players defect

	cooperate	defect	col payoff ↓ (a, b) ↑ row's payoff
cooperate	$(-1, -1)$	$(-10, 0)$	
defect	$(0, -10)$	$(-5, -5)$	