Two-player Zero-Sum Games

Announcements

- Midterm on Thursday!
- Review session today 4-6
- No lecture Thursday
- No discussions this week

Zero-sum games

Input: A payoff matrix M

Row player: picks row r
Col player: picks col c

$$\left.\begin{array}{l}\text{Row player: picks row } r\\ \text{Col player: picks col } c\end{array}\right\} \rightarrow \left\{\begin{array}{l}\text{Win } +A[r,c]\\ \text{Win } -A[r,c]\end{array}\right.$$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Types of strategies:

"Pure strategy": a single row/column

e.g. row player always picks rock (beaten by paper col)

"Mixed strategy": probability distribution over pure strategies

e.g. $\Pr[\text{Rock}] = 1/3$, $\Pr[\text{Paper}] = 1/3$, $\Pr[\text{Scissors}] = 1/3$

Note: average score $= 0$ (no matter what col player does)

Game 1

Turn Order:

1. Row player announces mixed strat $P = (p_1, p_2)$
2. Col player responds w/ mixed strat $q = (q_1, q_2)$

$p_1 = \Pr[\text{Row } 1]$

$p_2 = \Pr[\text{Row } 2]$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 3 & -1 \\ 2 & -2 & 1 \\ \hline & q_1 & q_2 \end{array}$$

Def: Row player's average score is $\text{Score}(p,q) = 3 \cdot p_1 q_1 - 1 \cdot p_1 q_2 - 2 \cdot p_2 q_1 + 1 \cdot p_1 q_2$

Col player's best strat: $\underset{\text{mixed strat } q}{\text{minimize}} \left\{ \text{Score}(p,q) \right\}$

(should pick best col) $= \underset{\text{pure strat } q}{\min} \left\{ \text{Score}(p,q) \right\} = \min \{ \underbrace{3 \cdot p_1 - 2 \cdot p_2}_{\text{col 1 score}}, \underbrace{-1 \cdot p_1 + 1 \cdot p_2}_{\text{col 2 score}} \}$

Row player's best strat: $\underset{\text{mixed strat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1, -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$

Fact: Can calculate $\underset{\text{mixed strat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1 ,\ -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$ with LP.

Pf: maximize z

subject to

$$z \leq 3p_1 - 2p_2$$

$$z \leq -p_1 + p_2$$

$$p_1 + p_2 = 1$$

$$p_1 \geq 0,\ p_2 \geq 0.$$

$= LP_1$

Note: $z = \min\{3p_1 - 2p_2, -p_1 + p_2\}$. $\square$

Game 2

Same as Game 1, except col player goes 1st and row player goes 2^{nd}

	1	2	
1	3	−1	P_1
2	−2	1	P_2
	q_1	q_2	

Payoff of row 1: $3 \cdot q_1 - 1 \cdot q_2$

Payoff of row 2: $-2 \cdot q_1 + 1 \cdot q_2$

Row player's best strat $= \max \{3q_1 - q_2, -2q_1 + q_2\}$

Col " " " $= \underset{\text{mixed strat } q}{\text{minimize}} \{ \max \{3q_1 - q_2, -2q_1 + q_2\}\}$

$LP_2 =$ minimize Z,

subject to
$$3q_1 - q_2 \le Z$$
$$-2q_1 + q_2 \le Z$$
$$q_1 + q_2 = 1, \quad q_1 \geq 0, q_2 \geq 0.$$

Game 1

1. Row player first
2. Col player second

$$\begin{array}{c|c|c|}
 & 1 & 2 \\
\hline
1 & 3 & -1 \\
\hline
2 & -2 & 1 \\
\hline
\end{array}$$

Game 2

1. Col player first
2. Row player second

$$\max_{p}\left\{\min_{q}\left\{\text{Score}(p,q)\right\}\right\} \leq \min_{q}\left\{\max_{p}\left\{\text{Score}(p,q)\right\}\right\}$$

$$\begin{array}{ccc}
\| & & \| \\
LP_1 & \xleftrightarrow{\text{(dual)}} & LP_2
\end{array}$$

$\therefore$ Strong duality $\Rightarrow$ $LP_1 = LP_2 = \text{Value(Game)}$ (Definition)

(Min-Max Theorem)

$\Rightarrow$ order of play doesn't change value

$\Rightarrow$ $\exists$ optimal strat for ROW, irrespective of COL

Minimax strats

$$\begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & 3 & -1 & p_1 \\ 2 & -2 & 1 & p_2 \\ \hline & q_1 & q_2 & \end{array}$$

Row player's optimum strategy

$p_1 = 3/7 \qquad p_2 = \frac{4}{7}$

Payoff of col 1 $= \frac{3}{7} \cdot 3 + \frac{4}{7} \cdot (-2) = \frac{1}{7}$

Payoff of col 2 $= \frac{3}{7} \cdot (-1) + \frac{4}{7} \cdot 1 = \frac{1}{7}$ (doesn't matter what col player does!)

Col player's optimum strategy

Value(Game) $= \frac{1}{7}$

$q_1 = \frac{2}{7} \qquad q_2 = \frac{5}{7}$

Payoff of row 1 $= \frac{2}{7} \cdot 3 + \frac{5}{7} \cdot (-1) = \frac{1}{7}$

Payoff of row 2 $= \frac{2}{7} \cdot (-2) + \frac{5}{7} \cdot (1) = \frac{1}{7}$ (doesn't matter what row player does!)

(Similar to RPS strategy w/ $Pr[\text{Rock}] = Pr[\text{Paper}] = Pr[\text{Scissors}]$)
$= 1/3$

Example problem

Suppose row player goes 1^{st} and plays

$P_1 = \frac{1}{2}$ $\quad P_2 = \frac{1}{5}$ $\quad P_3 = \frac{3}{10}$

P_1	X	10	20
P_2	40	30	10
P_3	20	40	10

If the col player's optimal response is not unique, what is X ?

Extra stuff

Def: **Nash equilibrium** = pair of mixed strats (p, q) s.t. if row player plays p, col player plays q, neither player has incentive to deviate

In 2PZSG, Nash equilibria = minimax strats

Non zero-sum game

"Prisoner's dilemma"

	cooperate	defect
cooperate	$(-1,-1)$	$(-10,0)$
defect	$(0,-10)$	$(-5,-5)$

col payoff

(a, b)

row's payoff

Nash equilibria = both players defect