Continue Linear Programming. Applications.
Lecture in a minute.

What’s a linear program?
Variables.
Linear inequalities, and a linear objective function.
Geometrically: a convex region in \( n \).
Optimal solution at “vertex” of region.
Cartoon simplex/duality: move to better vertex, repeatedly.

Applications:
Production Planning.
Variables hiring/firing/inventory/production.
Constraints/Objective encode costs and resource limits.
Bandwidth Problem.
Variables for routes.
Constraints/Objective encode revenue and resource limits.

Linear Programs.
Types of constraints: equality.
Non-negative versus unrestricted.
Standard Form.
Matrix, vector Notation.
Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas get 3 sq. yards/bushel of sunny land.
Carrots get 3 sq. yards/bushel of shady land.

Garden has 60 sq. yards of sunny land and 75 sq. yards of shady land.

To pea or not to pea, that is the question!
To pea or not to pea.

4$ for peas. 2$ bushel of carrots. $x_1$- to pea! $x_2$ carrots
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Peas take 3 unit of water/bushel.
Carrot take 2 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 3 yards/bushel of sunny land. Have 60 sq. yards.
$3x_1 \leq 60$
Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.
$3x_2 \leq 75$
Can’t make negative! $x_1, x_2 \geq 0$.

A linear program.

$$\max 4x_1 + 2x_2$$

$$3x_1 \leq 60$$

$$3x_2 \leq 75$$

$$3x_1 + 2x_2 \leq 100$$

$$x_1, x_2 \geq 0$$
Optimal Point?

\[
\begin{align*}
\max & \quad 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal point?
Try every point if we only had time!
How many points?
Real numbers?
Infinite. Uncountably infinite!
Where’s Waldo?

A linear program.

$$\text{max } 4x_1 + 2x_2$$

$$3x_1 \leq 60$$

$$3x_2 \leq 75$$

$$3x_1 + 2x_2 \leq 100$$

$$x_1, x_2 \geq 0$$

Optimal point?
Feasible Region.

Convex.

Any two points in region connected by a line in region. Algebraically:
If \( x \) and \( x' \) satisfy constraint, so does \( x'' = \alpha x + (1 - \alpha)x' \)
E.g. \( 3x \leq 60 \) and \( 3x' \leq 60 \)
\[ \rightarrow 3\alpha x \leq \alpha(60) \text{ and } 3(1 - \alpha)x' \leq (1 - \alpha)60 \]
\[ \rightarrow 3(\alpha(x) + (1 - \alpha)x') \leq (\alpha + (1 - \alpha))60 = 60 \]
Vertex is a solution.

“Isocline” - all points have same value on hyperplane.

Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints! Which are lines in 2 dimensions!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables (dimensions), $m$ constraints, how many?
$nm$? $\binom{m}{n}$? $n + m$?
$n$ constraints define point ....so $\binom{m}{n}$ possible vertices.
Finite!!!!!!! But exponential in the number of variables.
Simplex in 2 dimensions.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]

Simplex: Start at vertex. Move to better neighboring vertex. Until you stop. This example.

(0,0) objective 0. → (0,25) objective 50.
→ (16\(\frac{2}{3}\),25) objective 116\(\frac{2}{3}\) → (20,20) objective 120.

Duality:
Combine blue equations to upper bound objective function?
1/3 times first plus 1 times the third.
Get \(4x_1 + 2x_2 \leq 120\). Every solution must satisfy this inequality!
Objective value: 120.
Can we do better? Yes? No? Maybe? No! There is a solution.

Dual problem: add equations to get best upper bound.
More variables.

More vegetables. How about some Kale!
3$ per bushel.
2 sq. yards/bushel per sunny or 3 sq. yards/bushel for shady land.
2 units of water.

$x_3$ - sunny kale $x_4$ - shady kale.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 + 3x_3 + 3x_4 \\
3x_1 + 2x_3 & \leq 60 \\
3x_2 + 3x_4 & \leq 75 \\
3x_1 + 2x_2 + 2x_3 + 2x_4 & \leq 100 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]
Carpet production planning.

Demands: \(d_1, d_2, \ldots, d_{12}\), range: 440 – 920
30 employees. 20 carpets/month. 2000/month.
Overtime: 80% extra. Also at most 30% for one employee.
Hiring/firing: 320/400.
Storage: 8/carpet and no storage at the end of year.

Variables.
\(w_i\) - workers in month \(i\); 
\(w_0 = 30\)
\(x_i\) - carpets made in month \(i\)
\(o_i\) - overtime carpets in month \(i\)
\(h_i, f_i\) - hired/fired in month \(i\)
\(s_i\) - stored at end of month \(i\);
\(s_{12} = 0\)

Nonnegative: \(w_i, x_i, o_i, h_i, f_i, s_i \geq 0\)
Production: \(x_i = 20w_i + o_i\)
Employment: \(w_i = w_{i-1} + h_i - f_i\)
Inventory: \(s_i = s_{i-1} + x_i - d_i\)
Regulations: \(o_i \leq 6w_i\)
Objective:
\[
\min \quad 2000 \sum w_i + 320 \sum h_i + 400 \sum f_i + 8 \sum s_i + 180 \sum o_i.
\]
Problem:

$A − B$ pays 3$ per unit,
$A − C$ pays 2$ per unit,
$B − C$ pays 4$ per unit.

Every pair gets 2 units.

Linear Program Variables/Constraints:

$X_{AB}$ - flow along $A − a − b − B$.

$X'_{AB}$ is flow along path $A − a − c − b − B$

Capacity constraint on edge $(a, b)$:

$X_{AB} + X'_{AC} + X'_{AC} \leq 6$

Bandwidth constraint:

$X_{AB} + X'_{AB} \geq 2$


How many bandwidth constraints? 3.

Objective function?

$$3(X_{AB} + X'_{AB}) + 4(X_{BC} + X'_{BC}) + 2(X_{AC} + X'_{AC})$$
Again with carpets!

Production: $x_i = 20w_i + o_i$
Employment: $w_i = w_{i-1} + h_i - f_i$
Inventory: $s_i = s_{i-1} + x_i - d_i$
Regulations: $o_i \leq 6w_i$

$$\min 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum i s_i + 180 \sum i o_i.$$  

Different form!

Not for example: $x_1 + x_2 \leq 7.$
Variants of linear programs.

1. Maximization or minimization.
2. Equations or inequalities.
3. Non-negative variables or unrestricted variables.
Translations/Reductions.

1. Maximization to minimization?
   Multiply objective function by $-1$.

2. Less than inequalities into greater than?
   Multiply both sides by (-1) again!
   Example: $4 \geq 3$ to $(-1)4 \leq (-1)3$.

3. Inequalities and equalities.
   (a) $\sum a_i x_i \leq b$ into equality?
   $\sum a_i x_i + s = b$ and $s \geq 0$.
   (b) $\sum a_i x_i = b$ into inequalities?
   $\sum a_i x_i \leq b$ and $\sum a_i x_i \geq b$

4. Simulate unrestricted variable $x$ with positive variables.
   - Introduce $x_+$, and $x_-$.
   - Replace $x$ by $(x_+ - x_-)$.

$(x_+ - x_-)$ could be any real number!
Standard Form.

Standard form. Minimization, positive variables, and “greater than” inequalities.

Peas and carrots.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
& 2x_1 \leq 60 \\
& 3x_2 \leq 75 \\
& 3x_1 + 2x_2 \leq 100 \\
& x_1, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min } & -4x_1 - 2x_2 \\
& -2x_1 \geq -60 \\
& -3x_2 \geq -75 \\
& -3x_1 - 2x_2 \geq -100 \\
& x_1, x_2 \geq 0
\end{align*}
\]
Matrix Form.

Recall Linear equations: \( Ax = b \)?
Can do that here, too!

\[
\begin{align*}
\min -4x_1 - 2x_2 \\
-2x_1 & \geq -60 \\
-3x_2 & \geq -75 \\
-3x_1 - 2x_2 & \geq -100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\min \begin{bmatrix} -4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \geq \begin{bmatrix} -60 \\ -75 \\ -100 \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \geq 0
\end{align*}
\]

Inputs:
\( m \times n \) matrix \( A \); \( m \) length vector \( b \); \( n \) length vector \( c \).
Output: \( n \) length vector \( x \).

\[
\begin{align*}
\min cx \\
Ax & \geq b
\end{align*}
\]
Linear Program Problem

Inputs:
$m \times n$ matrix $A$; $m$ length vector $b$; $n$ length vector $c$.
Output: $n$ length vector $x$.

$$\min cx$$
$$Ax \geq b$$

Oh yes, some complexities here.

1. Program has constraints $x_1 \leq 1$ and $x_1 \geq 3$?
   Has no feasible solution!
   **Infeasible.**

2. Program $x_1 \geq 0$, $\max x_1$.
   Optimum?
   100, 200, 300 ... no limit!
   **Unbounded.**
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