P and NP

So far ... Integer multiplication $O(n \log n)$

Minimum Spanning Trees $O((n+m)\log(n))$

All pairs shortest paths $O(n^3)$

...

Def: A problem is efficiently solvable if it can be solved in polynomial time $= O(n^k)$

(in theory; in practice, efficient can even mean $O(n)$ only)

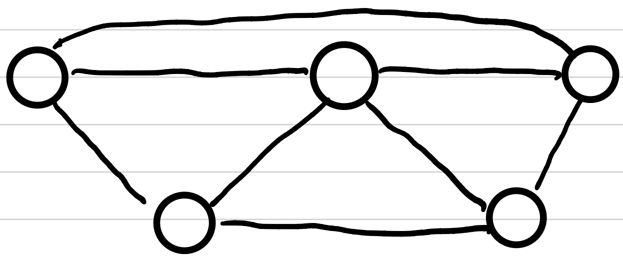
Def: P = "complexity class" of all problems which are efficiently solvable

Def: NP = class of problems whose solutions can be verified efficiently

3-coloring

Input: Graph $G=(V,E)$

Solution: A coloring $c: V \to \{\text{Red}, \text{Green}, \text{Blue}\}$ s.t. each edge receives 2 different colors



Factorization

Input: n-bit number N

Solution: two numbers $p, q > 1$ s.t. $p \cdot q = N$

Rudrata Cycle aka Hamiltonian Cycle

Input: Graph $G=(V,E)$

Solution: tour = cycle $v_1, v_2, \ldots, v_n, v_1$ visiting every node exactly once

Trivial alg: Try all $n!$ ways of ordering vertices

Best known alg: Time $O(1.657^n)$

Not known to be in P!

Fact: Hamiltonian Cycle $\in$ NP.

Pf: Verify(Input G, Solution tour $v_1, v_2, \ldots, v_n, v_1$): Check that
- visits each vertex once
- $\forall\, i,\ (v_i, v_{i+1}) \in E$

Eulerian cycle: find cycle visiting each edge exactly once

in P!

Traveling Salesperson Problem (TSP)

Input: Graph $G=(V,E)$ w/ edge weights
Solution: tour w/ low total weight

Optimization version: Min-TSP
Find the tour w/ min total weight.

Best known alg: time $O(n^2 2^n)$

Min-TSP $\in$ NP??? ⟵ Probably not!

Search version: Search-TSP
Find a tour w/ total weight $\leq B$ ⟵ "Budget" (part of input)

Fact: Search TSP $\in$ NP

Decision version: Decision-TSP
Does there exist a tour w/ weight $\leq B$? (Yes/No answer)

Fact: Dec-TSP $\in$ NP

Super formally

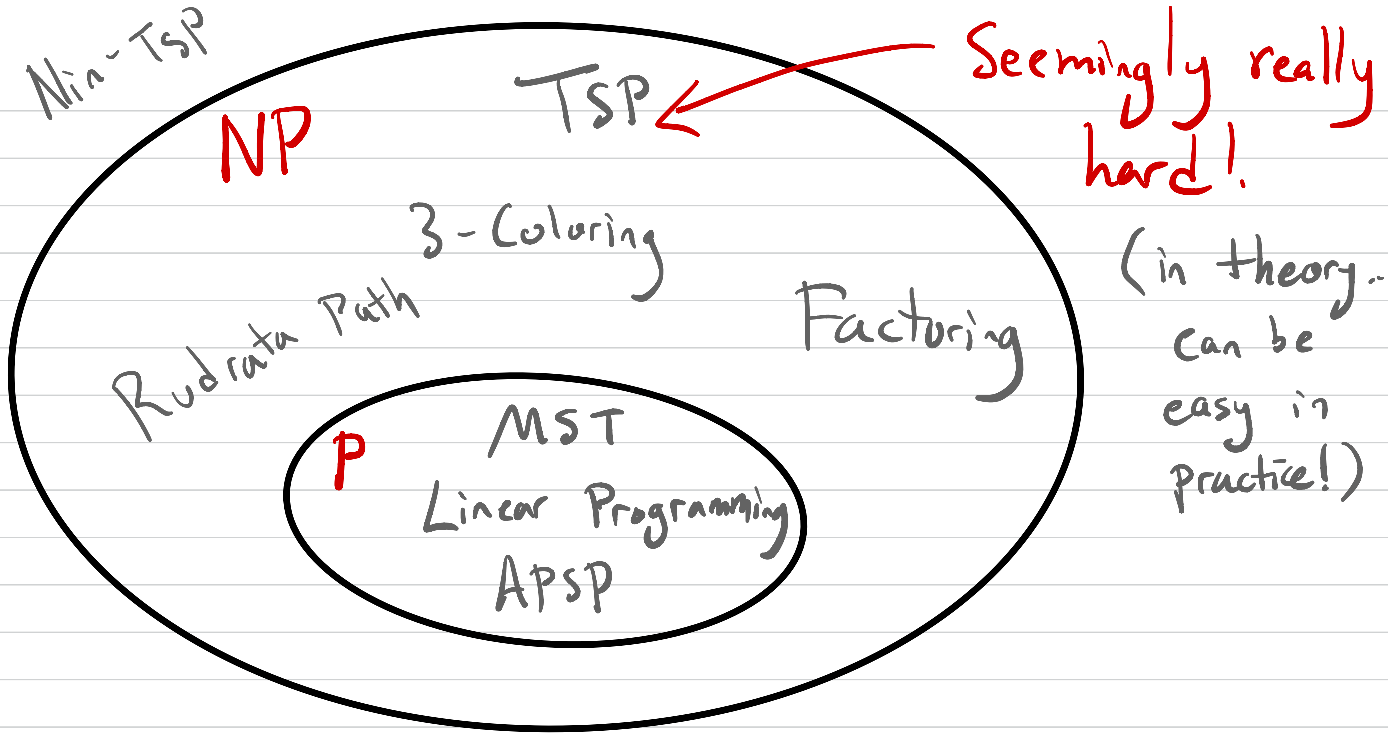
Def: A binary relation is a subset $R \subseteq \{0,1\}^* \times \{0,1\}^*$
If $(I,S) \in \{0,1\}^* \times \{0,1\}^*$, I = Instance, S = Solution

Problems:
- Verify(R): given (I,S), is $(I,S) \in R$?
- Search(R): given I, return S s.t. $(I,S) \in R$
  or "no solution" if none exists
- Decide(R): given I, output "yes" if exists solution
  "no" if none exists

Def: NP = all R which can be verified in time poly(n), $n = |I|$
P = all R in NP for which Search(R) can be solved in time poly(n)

Disclaimer: These are nonstandard definitions of P and NP.
In the real world, these are sometimes called FP and FNP.
P and NP typically defined for decision problems. (CS172)

Things not believed to be in NP:
1. Optimization versions of some problems
2. Really, really hard problems (Halting)



Biggest open problem in TCS: Is P = NP?

Q: How to compare difficulty of solving problems? A: Reductions