3.36pt

CS170: Lecture 2

Last Time: Place value is democratizing!
Like the printing press!
Reading, writing, arithmetic!
Input size/representation really matters!

Today: Chapter 2.
Divide and Conquer $\equiv$ Recursive.

Lecture in one minute!

Integer Multiplication: Gauss plus recursion is magic!
$O(n^2) \to O(n^{\log_2 3}) \approx O(n^{1.58..})$
Double size, time grows by a factor of 3.

Master's theorem: understand the recursion tree!
$T(n) = aT(\frac{n}{b}) + f(n)$.
Branching by a
diminishing by b
working by $O(f(n))$.
Leaves: $n^{\log_b a}$, Work: $\sum_i a^i f(\frac{n}{b^i})$.

Recursive (Divide and Conquer) Matrix Multiplication:
8 subroutine calls of size $n/2 \times n/2$
$\to O(n^3)$.
Strassen:
7 subroutine calls of size $n/2 \times n/2$
$\to O(n^{\log_2 7}) \approx O(n^{2.8})$.

Chapter 2.

Divide and conquer.

Definition of Multiplication.

n-bit numbers: x, y.

$$\begin{array}{r} \boxed{x_k x_{k-1} \quad x \quad\quad} \\ \times \boxed{\quad\quad y \quad y_1 y_0} \\ \hline \boxed{\quad\quad\quad xy \quad\quad\quad} \end{array}$$

kth "place" of xy: coefficient of 2^k:

$$a_k = \sum_{i \le k} x_i y_{k-i}.$$

$x * y = \sum_{k=0}^{2n} 2^k a_k$.

Number of "basic operations":

$$\sum_{k \le 2n} \min(k, 2n-k) = \Theta(n^2).$$

Recursive Algorithm for Multiplication.

Two n-bit numbers: x, y.

$$\begin{array}{lcl} x & = & \boxed{\quad x_L \quad}\boxed{\quad x_R \quad} = 2^{n/2} x_L + x_R \\ y & = & \boxed{\quad y_L \quad}\boxed{\quad y_R \quad} = 2^{n/2} y_L + y_R \end{array}$$

Multiplying out

$$\begin{aligned} x \times y &= (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R) \\ &= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R \end{aligned}$$

Four $n/2$-bit multiplications: $x_L y_L$, $x_L y_R$, $x_R y_L$, $x_R y_R$.
Recurrence:

$$T(n) = 4T(\frac{n}{2}) + O(n)$$

Recurrence for recursive algorithm.

Recurrence:

$$T(n) = 4T(\frac{n}{2}) + \Theta(n)$$

$T(n)$ is

(A) $\Theta(n)$.

(B) $\Theta(n^2)$.

(C) $\Theta(n^3)$.

Idea: Think about recursion tree.
A degree 4 tree of depth $\log_2 n$.
$4^{\log_2 n} = (2^2)^{\log_2 n} = 2^{2\log_2 n} = (2^{\log_2 n})^2 = n^2$
$\Theta(n^2)$ leaves or base cases.
One for each pair of digits!

Really? Unfolded recursion in my head?!?!
How did I really obtain bound? Soon a formula.

TBH,unfolded recurrence in head. Don't remember formulas.

Demo

As number of bits double:

Elementary School Multiply:

$O(n^2)$
$n \to 2n$

Runtime: $T = cn^2 \to T' = c(2n)^2 = 4(cn^2) = 4T$

Python multiply:

$n \to 2n$
Runtime: $T \to 3T$.

Asymptotics: $T = cn^w \to c((2n)^w) = T' = 3T = 3(cn^w)$.
.... $\to 2^w = 3$. or $w = \log_2 3 \approx 1.58$.

Python multiply: $O(n^{\log_2 3})$

Much better than grade school.

Multiply Complex Numbers

$(3+2\,\mathbf{i})(4+5\,\mathbf{i}) = 12 + (15+8)\,\mathbf{i} + 10\,\mathbf{i}^2$

Recall, $\mathbf{i}^2 = -1$, so simplifying

$(12-10) + 22\,\mathbf{i} = 2 + 22\,\mathbf{i}$.

What about $(32765 + 219898\,\mathbf{i})(413764 + 511110\,\mathbf{i})$?

Gauss's trick.

$$(a + b\,\mathbf{i})(c + d\,\mathbf{i}) = (ac - bd) + (ad + bc)\,\mathbf{i}.$$

Four multiplications: ac, bd, ad, bd.

Drop the i:

$$P_1 = (a+b)(c+d) = ac + ad + bc + bd.$$

Four multiplications from one! ..but all added up.

Two more multiplications: $P_2 = ac$, $P_3 = bd$.

$$(ac - bd) = P_2 - P_3.$$

$$(ad + bc) = P_1 - P_2 - P_3.$$

Only three multiplications. An extra addition though!
Which is harder of multiplication or addition?
Multiplication!

Faster Algorithm for Multiplication.

Two n-bit numbers: x, y.

$$x = 2^{n/2} x_L + x_R \quad ; \quad y = 2^{n/2} y_L + y_R$$
$$x \times y \quad = \quad 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

Need 3 terms: $x_L y_L$, $x_L y_R + x_R y_L$, $x_R y_R$.

Used four $\frac{n}{2}$-bit multiplications: $x_L y_L$, $x_L y_R$, $x_R y_L$, $x_R y_R$.

Can you compute three terms with 3 multiplications?

(A) Yes.

(B) No

(A) Yes.

Three multiplications and faster algorithm.

Two n-bit numbers: x, y.

$$x = 2^{n/2} x_L + x_R \quad ; \quad y = 2^{n/2} y_L + y_R$$
$$x \times y \quad = \quad 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

Need 3 terms: $x_L y_L$, $x_L y_R + x_R y_L$, $x_R y_R$.

Compute
$P_1 = (x_L + x_R)(y_L + y_R) = x_L y_L + x_L y_R + x_R y_L + x_R y_R$.

Two more: $P_2 = x_L y_L$, $P_3 = x_R y_R$. $(x_L y_R + x_R y_L) = P_1 - P_2 - P_3$
3 multiplications!

$$T(n) = 3T(\frac{n}{2}) + \Theta(n)$$

Technically: $\frac{n}{2} + 1$ bit multiplication. Don't worry.

Analysis of runtime.

Recurrence for "fast algorithm".

$$T(n) = 3T(\frac{n}{2}) + \Theta(n)$$

Runtime is

(A) $\Theta(n)$

(B) $\Theta(n^2)$

(C) $\Theta(n^{\log_2 3})$

(C) Idea: number of base cases is $n^{\log_2 3}$.

$3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n} = n^{\log_2 3}$

So multiplication algorithm with ..

$$T(n) = 3T(\frac{n}{2}) + \Theta(n) = \Theta(n^{\log_2 3}) = \Theta(n^{1.58...})!!!!$$

But: all digits have to multiply each other!

They do! $(a+b)(c+d) = ac + ac + bc + bd$

4 products from one multiplication!

Logarithms reminder.

Exponents Quiz: $(a^b)^c = (a^c)^b$?

Yes? No?

Yes. $(a^b)^c = a^{bc} = a^{cb} = (a^c)^b$.

Definition of log: $a = b^{\log_b a}$

Logarithm Quiz: $a^{\log_b n} = n^{\log_b a}$?

Yes!

$$a^{\log_b n} = (b^{\log_b a})^{\log_b n} = (b^{\log_b n})^{\log_b a} = n^{\log_b a}$$

Solving recurrences.

$$T(n) = 4T(\frac{n}{2}) + cn; \qquad T(1) = c$$

Recursion Tree	# probs	sz	time/prob	time/level
$T(n)$	1	n	cn	cn
$T(\frac{n}{2})\ T(\frac{n}{2})\ T(\frac{n}{2})\ T(\frac{n}{2})$	4	$\frac{n}{2}$	$c(\frac{n}{2})$	$2cn$
$T(\frac{n}{4})\cdots T(\frac{n}{4}) \quad T(\frac{n}{4})\cdots T(\frac{n}{4})$	4^2	$\frac{n}{4}$	$c(\frac{n}{4})$	$4cn$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	4^i	$\frac{n}{2^i}$	$c(\frac{n}{2^i})$	$2^i cn$

$\frac{n}{2^i} = 1$ when $i = \log_2 n \implies$ Depth: $d = \log_2 n$.

$4^{\log n} = 2^{2\log n} = n^2$ base case problems. size 1. Work/Prob: c

Work: cn^2.

Total Work: $cn + 2cn + 4cn + \cdots + cn^2 = O(n^2)$. Geometric series.

Solving recurrences.

$$T(n) = 4T(\frac{n}{2}) + cn; \qquad T(1) = c$$

Recursion Tree	# probs	sz	time/prob	time/level
$T(n)$	1	n	cn	cn
$T(\frac{n}{2})\ T(\frac{n}{2})\ T(\frac{n}{2})\ T(\frac{n}{2})$	4	$\frac{n}{2}$	$c(\frac{n}{2})$	$2cn$
$T(\frac{n}{4})\cdots T(\frac{n}{4}) \quad T(\frac{n}{4})\cdots T(\frac{n}{4})$	4^2	$\frac{n}{4}$	$c(\frac{n}{4})$	$4cn$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	4^i	$\frac{n}{2^i}$	$c(\frac{n}{2^i})$	$2^i cn$

$\frac{n}{2^i} = 1$ when $i = \log_2 n \implies$ Depth: $d = \log_2 n$.

$4^{\log n} = 2^{2\log n} = n^2$ base case problems. size 1. Work/Prob: c

Work: cn^2.

Total Work: $cn + 2cn + 4cn + \cdots + cn^2 = O(n^2)$. Geometric series.

Fast multiplication.

$$T(n) = 3T(\frac{n}{2}) + cn; \qquad T(1) = c$$

Recursion Tree	# probs	sz	time/prob	time/level
$T(n)$	1	n	cn	cn
$T(\frac{n}{2})\ T(\frac{n}{2})\ T(\frac{n}{2})$	3	$\frac{n}{2}$	$c(\frac{n}{2})$	$(\frac{3}{2})cn$
$T(\frac{n}{4})\cdots T(\frac{n}{4}) \quad T(\frac{n}{4})\cdots T(\frac{n}{4})$	3^2	$\frac{n}{4}$	$c(\frac{n}{4})$	$(\frac{3}{2})^2 cn$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	3^i	$\frac{n}{2^i}$	$c(\frac{n}{2^i})$	$(\frac{3}{2})^i cn$

$\frac{n}{2^i} = 1$ when $i = \log_2 n \implies$ Depth: $d = \log_2 n$.

$3^{\log_2 n} = n^{\log_2 3}$ base case problems. size 1. Work/Prob: c. Work: $cn^{\log_2 3}$.

Total Work: $cn + (\frac{3}{2})cn + \cdots + cn^{\log_2 3} = O(n^{\log_2 3})$ Geometric series.

Divide and Conquer: In general.

$$T(n) = aT(\frac{n}{b}) + O(n^d); \qquad T(1) = c$$

Recursion Tree	# probs	sz	time/prob	time/lvl
$T(n)$	1	n	cn^d	cn^d
$T(\frac{n}{b})\ T(\frac{n}{b})\ T(\frac{n}{b})$	a	$\frac{n}{b}$	$c(\frac{n}{b})^d$	$(\frac{a}{b^d})cn^d$
$T(\frac{n}{b^2})\cdots T(\frac{n}{b^2}) \quad T(\frac{n}{b^2})\cdots T(\frac{n}{b^2})$	a^2	$\frac{n}{b^2}$	$c(\frac{n}{b^2})^d$	$(\frac{a}{b^d})^2 cn^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	a^i	$\frac{n}{b^i}$	$c(\frac{n}{b^i})^d$	$(\frac{a}{b^d})^i cn^d$

$\frac{n}{b^i} = 1$ when $i = \log_b n \implies$ Depth: $k = \log_b n$.

Level i work: $(\frac{a}{b^d})^i n^d$.

Master's Theorem

Depth: $\log_b n$.
Level i work:

$$\left(\frac{a}{b^d}\right)^i n^d.$$

Total:

$$n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

Geometric series: If $\frac{a}{b^d} < 1$ ($d > \log_b a$), first term dominates

$$O(n^d),$$

if $\frac{a}{b^d} > 1$ ($d < \log_b a$), last term dominates.

$$O(n^{\log_b a}),$$

and if $\frac{a}{b^d} = 1$ ($d = \log_b a$), then all terms are the same

$$O(n^d \log_b n).$$

Master's Theorem: examples.

For a recurrence $T(n) = aT(n/b) + O(n^d)$
We have

$d > \log_b a \quad T(n) = O(n^d)$
$d < \log_b a \quad T(n) = O(n^{\log_b a})$
$d = \log_b a \quad T(n) = O(n^d \log_b n)$.

$T(n) = 4T(\frac{n}{2}) + O(n)$ $a = 4$, $b = 2$, and $d = 1$.
$d = 1 < 2 = \log_2 4 = \log_b a \implies T(n) = O(n^{\log_b a}) = O(n^2)$.

$T(n) = T(\frac{n}{2}) + O(n)$ $a = 1$, $b = 2$, and $d = 1$.
$1 > \log_2 1 = 0 \implies T(n) = O(n)$

$T(n) = 2T(\frac{n}{2}) + O(n)$ $a = 2$, $b = 2$, and $d = 1$.
$1 = \log_2 2 \implies T(n) = O(n \log n)$

Strassen

Matrix multiplication.

Strassen, 1968, visiting Berkeley.

Berkeley...Unite! Resist!

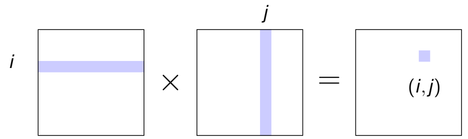
Strassen: Divide! conquer!

Matrix Multiplication

X and Y are $n \times n$ matrices.

$$Z = XY,$$

Z_{ij} is dot product of ith row with jth column.



$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}.$$

Runtime? $O(n^2)$? $O(n^3)$? n^2 entries in Z, $O(n)$ time per entry.
$O(n^3)$

Divide and Conquer

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$$

$A, B, C, \ldots, H$ are $\frac{n}{2} \times \frac{n}{2}$ matrices.

Subproblems?
$AE, BG, AF, BH, CE, DG, CF, DH$
are $n/2 \times n/2$ matrix multiplications.

Recurrence?

$$T(n) = 8T\left(\frac{n}{2}\right) + O(n^2).$$

8 subproblems, $O(n^2)$ to do the matrix additions.

Masters: $O(n^{\log_2 8}) = O(n^3)$.

Strassen

$P_1 = A(F - H) \quad P_5 = (A + D)(E + H)$
$P_2 = (A + B)H \quad P_6 = (B - D)(G + H)$
$P_3 = (C + D)E \quad P_7 = (A - C)(E + F)$
$P_4 = D(G - E)$

$$\begin{bmatrix} AE + BG = P_5 + P_4 - P_2 + P_6 & AF + BH = P_1 + P_2 \\ CE + DG = P_3 + P_4 & AF + BH = P_1 + P_5 - P_3 + P_7 \end{bmatrix}$$

$P_5 + P_4 - P_2 + P_6 =$
$(AE + AH + DE + DH) + (DG - DE) - AH - BH + BG + BH - DG - DH$
$= AE + BG$.

7 multiplies! Recurrence?

$T(n) = 7T(\frac{n}{2}) + O(n^2)$

From Masters:
(A) $O(n^2)$? (B) $O(n^{\log_2 7} \log n)$? (C) $T(n) = O(n^{\log_2 7})$?

Leaf subproblems dominate runtime!

(C) $O(n^{\log_2 7}) = O(n^{2.81\cdots})$ Way better than $O(n^3)$.

Commonly used in practice!

Current State of the Art: Matrix multiplication.

$k \times k$ multiplication in k^ω multiplications where $\omega = 2.37....$

E.g., Strassen: 2×2 multiplication in $2^{\log_2 7} = 7$ multiplications.

$T(n) = k^\omega T(\frac{n}{k}) + O(n^2)$

Masters: $O(n^{\log_k k^\omega}) = O(n^{\omega \log_k k}) = O(n^\omega)$

State of the art: k is very very large... e.g., 10^{100} ...but still a constant.

Based on complicated recursive constructions.

Improvement for constant + recursion gives better algorithm!

Example:
Gauss + recursion $\Longrightarrow$ faster multiplication.
Strassen's 7 multiplies + recursion $\Longrightarrow$ faster matrix multiplication.

Lecture in one minute!

Gauss plus recursion is magic!
$O(n^2) \to O(n^{\log_2 3}) \approx O(n^{1.58..})$
Double size, time grows by a factor of 3.

Master's theorem: understand the recursion tree!
Branching by a
diminishing by b
working by $O(f(n))$.
Leaves: $n^{\log_b a}$, Work: $\sum_i a^i f(\frac{n}{b^i})$.

Recursive (Divide and Conquer) Multiplication:
8 subroutine calls of size $n/2 \times n/2$
$\to O(n^3)$.
Strassen:
7 subroutine calls of size $n/2 \times n/2$
$\to O(n^{\log_2 7}) \approx O(n^{2.8})$.