CS 170: Algorithms

Standard Form: \( Ax \leq b, \max cx, x \geq 0 \)

Duality:

\[
\max x_1 + 8x_2 \\
x_1 \leq 4 \quad \quad \quad \quad \quad x_1 \leq 4 \\
x_2 \leq 3 \quad \quad \quad \quad \quad x_2 \leq 3 \\
x_1 + 2x_2 \leq 7 \quad \quad \quad (y_1 + y_2)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3 \\
x_1, x_2 \geq 0 \quad \quad \quad \quad \quad x_1, x_2 \geq 0
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives...
\( x_1 + 8x_2 \leq 4 + 24 = 28. \)
Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
\( x_1 + 8x_2 \leq 6(3) + 7 = 25. \)
Thus, the value is at most 25.
The upper bound is at most 25.
Proof of optimality!

Lecture in a minute.

Duality:

Primal: \( Ax \leq b, \max cx, x \geq 0 \)
Dual: \( A^Ty \geq b, \min by, y \geq 0 \)
Linear combination of equations that dominates objective function.
Duality: (typically) have the same value.
Weak Duality: \( \text{Primal} \leq \text{Dual}. \)
Feasible \( x, y \Rightarrow cx \leq y^TAx \geq y^Tb. \)

Simplex Implementation:
Start at a (feasible) vertex.
Begin at origin. Move to better neighboring vertex.
Coordinate system: distance from tight constraints.
Objective at origin in coordinate system.
\( O(mn) \) time to update linear system.
Until no better neighboring vertex.
Objective function in coordinate system is non-positive.
Dual Variables: new objective function!

Idea: Add up positive linear combination of inequalities to "get" upper bound on optimization function.
Will this always work?
How to find best upper bound?

Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>( x_1 \leq 4 )</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>( x_2 \leq 3 )</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>( x_1 + 2x_2 \leq 7 )</td>
</tr>
</tbody>
</table>

Adding equations thusly...
\( (y_1 + y_2)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3 \)
The left hand side should "dominate" optimization function:
If \( y_1, y_2, y_3 \geq 0 \)
and \( y_1 + y_2 \geq 1 \) and \( y_2 + 2y_3 \geq 8 \) then...
\( x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3 \)
Find best \( y_i 's \) to minimize upper bound?
Every solution must satisfy this inequality!

Duality: ... and dual is optimal!

The dual. 

In general.

Primal LP: 
\[
\begin{align*}
\text{max } & c \cdot x \\
\text{Ax } & \leq b \\
x \geq 0
\end{align*}
\]

Dual LP: 
\[
\begin{align*}
\text{min } & y^T b \\
y^T A & \geq c \\
y \geq 0
\end{align*}
\]

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal \( (P) \leq \) dual \( (D) \)

Feasible \((x, y)\)

\[
P(x) = c \cdot x \leq y^T Ax \leq y^T b = D(y).
\]

\( \implies P(x) \leq D(y) \).

Strong Duality: later.

Complementary Slackness

Primal LP: 
\[
\begin{align*}
\text{max } & c \cdot x \\
\text{Ax } & \leq b \\
x \geq 0
\end{align*}
\]

Dual LP: 
\[
\begin{align*}
\text{min } & y^T b \\
y^T A & \geq c \\
y \geq 0
\end{align*}
\]

Again: simplex

Simplex: Start at vertex. Move to better neighboring vertex.

Until no better neighbor. Duality:

Add blue equations to get objective function?

1/3 times first plus second.

Get \( 4x_1 + 2x_2 \leq 120 \). Every solution must satisfy this inequality!

Geometrically and Complementary slackness:

Add tight constraints to “dominate objective function.”

Don’t add this equation! Shifts.

Example: review.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x, x_2 & \geq 0
\end{align*}
\]

“Matrix form”

\[
\begin{align*}
\text{max } & [1, 8] \cdot [x_1, x_2] \\
\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \\
[x_1, x_2] & \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min } & 4y_1 + 3y_2 + 7y_3 \\
y_1 & \geq 1 \\
y_2 & \geq 3 \\
y_1 + 2y_2 & \leq 7 \\
x_1, x_2 & \geq 0 \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min } & [4.3, 7] \cdot [y_1, y_2, y_3] \\
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} & \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

We can rewrite the above in matrix form.

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad c = [1.8, 4.3, 7], \quad b = [4, 3, 7]
\]

The primal is \( Ax \leq b, max c \cdot x, x \geq 0 \).

The dual is \( y^T A \geq c, min b \cdot y, y \geq 0 \).
Solution(s)

\[
\begin{align*}
\max [1, 8] & \quad [x_1, x_2] \\
\min [4, 3, 7] & \quad [y_1, y_2, y_3] \\
\begin{bmatrix}
0 & 1 \\
1 & 2 \\
\end{bmatrix} & \begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} \leq \begin{bmatrix}
4 \\
7 \\
\end{bmatrix} \\
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} & \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{bmatrix} \geq \begin{bmatrix}
1 \\
2 \\
\end{bmatrix} \\
[1, 2] & \begin{bmatrix}
x_1, x_2 \\
y_1, y_2, y_3 \\
\end{bmatrix} \geq 0.
\end{align*}
\]

Primal: \((x_1, x_2) = (1, 3)\)
Feasible? \(1 \times 1 + 0 \times 3 \leq 4, 0 \times 1 + 1 \times 3 \leq 3, 1 \times 1 + 2 \times 3 \leq 7.\)
Value = \(1 \times 1 + 3 \times 8 = 25.\)
Dual: \((y_1, y_2, y_3) = (0.6, 1)\)
Feasible? \(1 \times 0 + 0 \times 6 + 1 \times 1 \geq 1, 0 \times 0 + 1 \times 1 + 2 \times 3 \geq 8.\)
Value = \(1 \times 1 + 3 \times 8 = 25.\)
Complimentary Slackness: \((b - Ax)(y) = 0.\)
Either slack for equation is 0 or dual variable is 0 or both.
First equation for primal: \(4 - (1 \times 1) + 0 \times 3 = 1\) and \(y_1 = 0.\)
In dual, both equations are tight. so both \(x_1\) and \(x_2\) can be non-zero in optimal.

Simplex Algorithm.

Start at a vertex. 
Move to better neighboring vertex. 
Until no better neighboring vertex.

\[
\begin{align*}
\max (x_1 + x_2) \\
7x_1 + 5x_2 \leq 21 \\
4x_1 + 5x_2 \leq 20 \\
2x_1 + 10x_2 \leq 33 \\
x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

Optimal value?

\[
\begin{align*}
x_1 = 1/3, x_2 = 3/7 \\
\text{Value is } 4/7.
\end{align*}
\]

Left hand sides: \(\frac{1}{15} (7x_1 + 5x_2) + \frac{2}{15} (4x_1 + 5x_2) = x_1 + x_2.\)
Right Hand Sides: \(\frac{2}{15} x_1 + \frac{4}{15} x_2 = 4/7\)
Value is no more than \(4/7.\)

Dual adds tight constraints to get objective function. 
Geometrically: can’t get better!

Dual.

\[
\begin{align*}
\max (x_1 + x_2) \\
7x_1 + 5x_2 \leq 21 \\
4x_1 + 5x_2 \leq 20 \\
2x_1 + 10x_2 \leq 33 \\
x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

Optimal value?

\[
\begin{align*}
x_1 = 1/3, x_2 = 3/7 \\
\text{Value is } 4/7.
\end{align*}
\]

Geometry of Dual.

Solution: \(x_1 = 1/3, x_2 = 3/7\)
Dual: \(y_1 = 1/15, y_2 = 2/15\)

Three dimensions.

Vertex?
Two tight constraints?
\(x_1 + x_2 + x_3 = 5\)
\(x_1 + 5x_2 + 2x_3 = 7\)
Which point?
Three unknowns, two equations.
Defines a line, not a point.
Three tight constraints define a vertex!
\(n\) dimensions \(\implies\) \(n\) variables \(\implies\) \(n\) constraints define vertex.
A constraint defines a hyperplane.
Line in two dimensions. Plane in three.
In \(n\) dimensions, vertex is intersection of \(n\) hyperplanes.

Test

\[
\begin{align*}
m \times n \text{ matrix } A. \text{ How many tight constraints at vertex?} \\
(A) \text{ At least } m. \\
(B) \text{ At most } n. \\
(C) \text{ At least } n. \\
\text{Dudette!} \\
\text{C. dimension of space is } n. \text{ } n \text{ constraints.} \\
\text{At least? } \text{May be redundant constraints!}
\end{align*}
\]
**Simplex Algorithm.**

Start at a vertex.
Move to better neighboring vertex.
Until no better neighboring vertex.

\[
\begin{align*}
\text{max}(x_1 + x_2) \\
7x_1 + 5x_2 &\leq 20 \\
4x_1 + 5x_2 &\leq 21 \\
2x_1 + 10x_2 &\leq 33 \\
x_1 &\geq 0, x_2 \geq 0
\end{align*}
\]

Why optimal? Draw line corresponding to \(cx = \) current value. Entire feasible region on "wrong" side.

---

**Simplex algorithm.**

Two tasks:
1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

\[
\begin{align*}
\text{max} c^T x \\
Ax &\leq b \\
x &\geq 0
\end{align*}
\]

Start at origin, supposing it is feasible.

At origin in new coordinate system!

New variables: \(y_1\) is distance from constraint \(i\).

Origin: feasible, value 0.
Inequalities 4 and 5 are tight.
Relax constraint \(x_2 = 0\).
Increase \(x_2\) until ...

**Example.**

\[
\begin{align*}
\text{max} 2x_1 + 5x_2 \\
2x_1 - x_2 &\leq 4 \\
x_1 + 2x_2 &\leq 9 \\
-x_1 + x_2 &\leq 3 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{align*}
\]

Rewrite linear program.

\[
\begin{align*}
\text{max} 2x_1 + 5x_2 \\
2x_1 - x_2 &\leq 4 \\
x_1 + 2x_2 &\leq 9 \\
-x_1 + x_2 &\leq 3 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{align*}
\]

Rewrite linear program with new coordinates.

\[
\begin{align*}
\text{max} 2(y_1) + 5(3 - y_2 + y_1)
\end{align*}
\]

- New variables: \(y_1 = x_1, y_2 = 3 + x_1 - x_2\).
- Solve for \(x_1\) and \(x_2\): \(x_1 = y_1\) and \(x_2 = 3 - y_2 + y_1\).
- Plug in for \(x_1\) and \(x_2\): objective function
  \[
  \text{max} \quad 2y_1 + 5y_2
  \]

**Going to a better place.**

Two tasks:
1. Check optimality of vertex?
2. Where to go next?

At origin, there is positive \(c\), so increase \(x_1\).

...until you hit another constraint.

\(x_1 \geq 0\) is no longer tight, but new constraint is. \(\Rightarrow n\) constraints!

At vertex!

**A new coordinate system.**

New coordinates: Distance from new tight constraints.

Constraint 1

Constraint 2

\(y_i\) is distance from constraint \(i\).

\(x\) is at \((y_1, y_2)\) in new coordinate system.

For constraint \(i\): \(y_i = b_i - a_i x\)

Recall that for origin: \(x_i\) was distance from constraint \(x_i \geq 0\).

At origin in new coordinate system!
Rewriting example..

\[
\begin{align*}
\text{max} & \quad 15 + 7y_1 - 5y_2 \\
y_1 + y_2 & \leq 7 \quad (1) \\
3y_1 - 2y_2 & \leq 3 \quad (2) \\
y_2 & \geq 0 \quad (3) \\
y_1 & \geq 0 \quad (4) \\
-y_1 + y_2 & \leq 3 \quad (5)
\end{align*}
\]

Rewrite: 
\[
\begin{align*}
z_2 &= y_2 \\
z_1 &= -3y_1 + 2y_2 - y_1 = -\frac{1}{2}y_1 + \frac{3}{2}z_2 + 1
\end{align*}
\]

Objective function.
\[
\begin{align*}
\text{max} & \quad 15 + 7y_1 - 5z_2 \\
y_1 + y_2 & \leq 7 \quad (1) \\
3y_1 - 2y_2 & \leq 3 \quad (2) \\
y_2 & \geq 0 \quad (3) \\
y_1 & \geq 0 \quad (4) \\
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Rewriting example..

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y_1 & \geq 0 \quad (4) \\
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\end{align*}
\]

Rewrite: 
\[
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z_2 &= y_2 \\
z_1 &= -3y_1 + 2y_2 - y_1 = -\frac{1}{2}y_1 + \frac{3}{2}z_2 + 1
\end{align*}
\]

Objective function.
\[
\begin{align*}
\text{max} & \quad 15 + 7(\frac{1}{2}z_1 + \frac{3}{2}z_2 + 1) - 5z_2 \\
\text{max} & \quad 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2
\end{align*}
\]

Optimal? Optimal point! Increasing \(z_1, z_2\) makes things worse.

Rewriting example..

\[
\begin{align*}
\text{max} & \quad 15 + 7y_1 - 5y_2 \\
y_1 + y_2 & \leq 7 \quad (1) \\
3y_1 - 2y_2 & \leq 3 \quad (2) \\
y_2 & \geq 0 \quad (3) \\
y_1 & \geq 0 \quad (4) \\
-y_1 + y_2 & \leq 3 \quad (5)
\end{align*}
\]

Rewrite: 
\[
\begin{align*}
z_2 &= y_2 \\
z_1 &= -3y_1 + 2y_2 - y_1 = -\frac{1}{2}y_1 + \frac{3}{2}z_2 + 1
\end{align*}
\]

Objective function.
\[
\begin{align*}
\text{max} & \quad 15 + 7(\frac{1}{2}z_1 + \frac{3}{2}z_2 + 1) - 5z_2 \\
\text{max} & \quad 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2
\end{align*}
\]

Optimal? Yes! Maybe not! Optimal point!

Review.

In each step:
LP in coordinate system from tight constraints.
Optimal?
Does objective function have nonnegative multiplier?
\[
\text{max} 15 + 7y_1 - 5y_2.
\]

Go to tight constraint along improving coordinate.
\[
3y_1 - 2y_2 \leq 3.
\]

Express LP in coordinate system for new tight constraints. See previous slides!
Repeat.

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An \(x\) where \(Ax \leq b\) and at vertex.

Make a new linear program.

Introduce positive variables \(z_i\) for inequality \(i\).

Constraints: \(a_i x - z_i \leq b_i\).

\[
\text{max} \sum z_i
\]

Vertex solution \((x, z)\) of value zero

\[
\Rightarrow \text{all } z\text{'s are zero} \\
\Rightarrow \text{all inequalities are satisfied} \\
\Rightarrow x \text{ is a feasible vertex of } Ax \leq b.
\]

Degeneracy.

Degenerate vertices. Intersection of more than \(n\) constraints.

Feasible

Problem: all neighboring vertices are no better.

Infinite looping: Bland's anticycling rule.

Or Perturb problem a bit. Unlikely to intersect!

Unboundedness.

Unbounded value.

Simplex can tell difference.

From \(X\): either unbounded improvement or optimal.
Running Time

Check optimality? $O(n)$.
Find tight constraint:
$O(m)$ constraints. $O(1)$ time per constraint.
$O(m)$ total.
Find new coordinate system, rewrite LP.

Recall $y_i = b_i - a_i x$

Rewrite in terms of $y_i$.

Solve for $x_i$ in terms of $y_i$.

Plug in.

Naively: $O(n^3)$ time.

Only one new constraint. One new $y_i$.

Only one unknown.

Backsolve.

$O(nm)$ time to update LP.

How many steps?

Could be large. Exponential in worst case!

Fast, in practice!

Lecture in a minute.

Duality:

Primal: $Ax \leq b$, $\max cx$, $x \geq 0$

Dual: $A^T y \geq b$, $\min by$, $y \geq 0$

Linear combination of equations that dominates objective function.

Duality: (typically) have the same value.

Weak Duality: Primal $\leq$ Dual.

Feasible $x, y \implies cx \leq y^T Ax \geq y^T b$.

Simplex Implementation:

Start at a (feasible) vertex.

(defined by linear system $A'x = [b, 0, \ldots, 0]$).

Begin at origin. Move to better neighboring vertex.

Coordinate system: distance from tight constraints.

Vertex at origin in coordinate system.

$O(nm)$ time to update linear system.

Until no better neighboring vertex.

Objective function in coordinate system is non-positive.

Dual Variables: new objective function!