Approximation Algorithms

Suppose you show your problem P is NP-hard. What now?

1. Learn more about inputs
2. Heuristic
3. Approximation algorithm $\longleftarrow$ today

Def: For a minimization problem P, an algorithm A is an α-approximation algorithm if for all instances I of P,

$$A(I) \leq \alpha \cdot OPT \qquad (\alpha \geq 1)$$

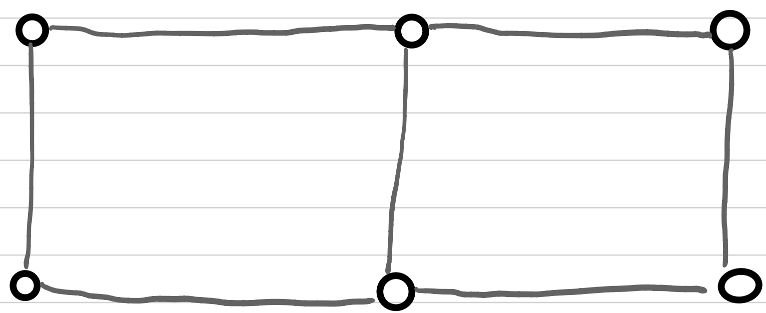
(maximization:) $A(I) \geq \alpha \cdot OPT \qquad (0 \leq \alpha \leq 1)$

Vertex Cover

Input: Graph $G=(V,E)$

Solution: A vertex cover $C \subseteq V$ of minimal size

Def: Vertex Cover = set of vertices s.t. every edge (u,v) is incident to one of them



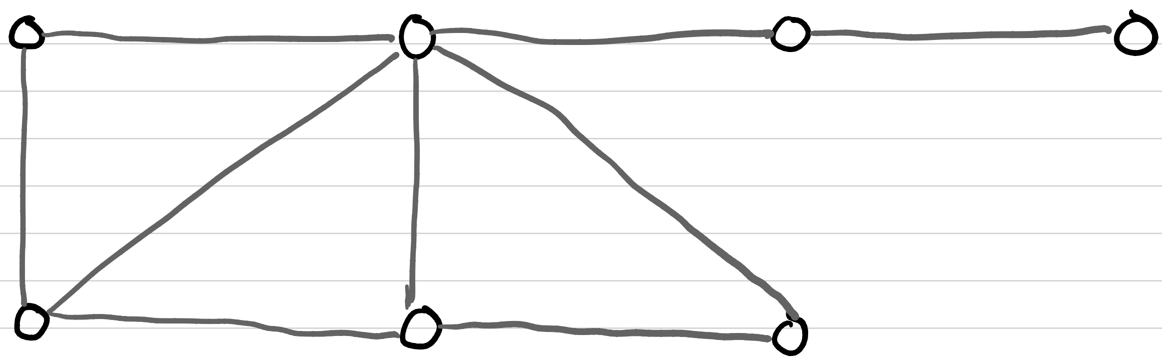
Thm: Vertex Cover is NP-hard (prove at home)

Algorithm #1

1. Compute a maximal matching M in G

Def: A matching is a set of edges w/ no overlapping vertices.
It is maximal if no more edges can be added to it.

Can compute by adding edges greedily to M.



2. Output $C = \{$ Both endpoints of all edges in $M \}$

Thm: C is a vertex cover and $|C| \leq 2 \cdot OPT$.

Claim: C is a vertex cover.

Pf: AFSOC that C is **not** a vertex cover.

Then exists $\underset{u}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{v}{\bullet} \in E$ s.t. $u, v \notin C$.

Then could add (u,v) to M. But M is maximal! Contradiction.

$\square$

Claim: $|C| \leq 2 \cdot OPT$, OPT = size of minimum vertex cover.

Pf: Any vertex cover must cover each edge $(u,v) \in M$

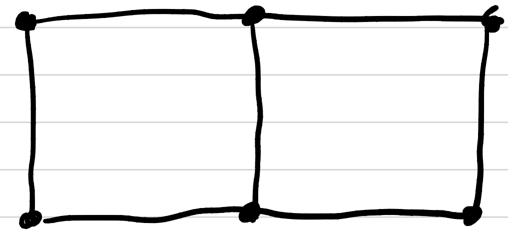
$\Rightarrow$ must include either u or v (or both)

$\therefore OPT \geq |M|$

So $|C| = 2 \cdot |M| \leq 2 \cdot OPT$. $\square$

Algorithm # 2: LP

Variable x_i for each vertex i



(ideally) $x_i = \begin{cases} 1 & \text{if } i \in \text{vertex cover} \\ 0 & \text{otherwise} \end{cases}$

Constraints: $0 \le x_i \le 1$

$x_i + x_j \geq 1 \quad \forall \; \underset{i}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\bullet} \in E$

Objective: $\min \sum_i x_i$

Claim: LP-OPT $\le$ OPT Vertex Cover

Pf: OPT Vertex Cover is feasible solution to LP. But also fractional solutions! □

"Alg": 1. Solve LP to get optimal solution $\{x_i^*, i = 1, \ldots, n\}$.

2. Convert x^* to a real vertex cover. "Rounding"

Rounding the LP

- Let $\{x_i^* \mid i = 1 \dots n\}$ be the optimal LP solution
- Rounding rule: $\left\{ \begin{array}{l} x_i^* \geq \frac{1}{2} \\ x_i^* < \frac{1}{2} \end{array} \right\} \begin{array}{l} \rightarrow \text{include in } C \\ \rightarrow \text{do not include in } C \end{array}$

Claim: C is a vertex cover

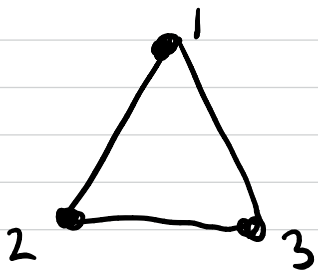
Pf: $i \bullet\!\!-\!\!-\!\!-\!\!\bullet j \Rightarrow x_i^* + x_j^* \geq 1 \Rightarrow$ either $x_i^* \geq \frac{1}{2}$ or $x_j^* \geq \frac{1}{2} \Rightarrow i \in C$ or $j \in C$. $\square$

Claim: $|C| \leq 2 \cdot OPT$.

Pf: $\forall\, i \in C$, Algorithm pays 1 unit

LP pays $x_i^* \geq \frac{1}{2}$

$\Rightarrow$ total cost of Alg $= |C| \leq 2 \cdot \text{LP-OPT} \leq 2 \cdot OPT$. $\square$



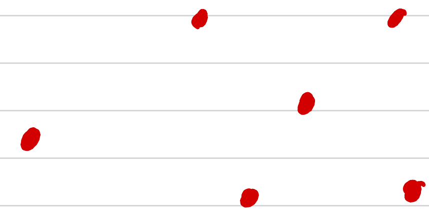
$$\min \quad x_1 + x_2 + x_3$$
$$\text{s.t.} \quad x_1 + x_2 \geq 1$$
$$x_2 + x_3 \geq 1$$
$$x_3 + x_1 \geq 1$$
$$0 \leq x_1, x_2, x_3 \leq 1$$

$$x_1^* = x_2^* = x_3^* = 1/2 \Rightarrow \text{LP-OPT} = x_1^* + x_2^* + x_3^* = \frac{3}{2}$$

Optimal Vertex Cover $= |\{1, 2\}| = 2$

Alg output $= \{1, 2, 3\} = 3$

(Metric) Traveling Salesperson Problem

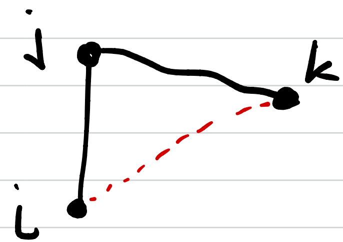


Input: • n cities
• pairwise distance d_{ij} $\forall\, i \neq j$

Solution: Minimum distance tour visiting every node exactly once

Metric assumption: triangle inequality

$$\forall\, i,j,k \quad d_{ij} + d_{jk} \geq d_{ik}$$

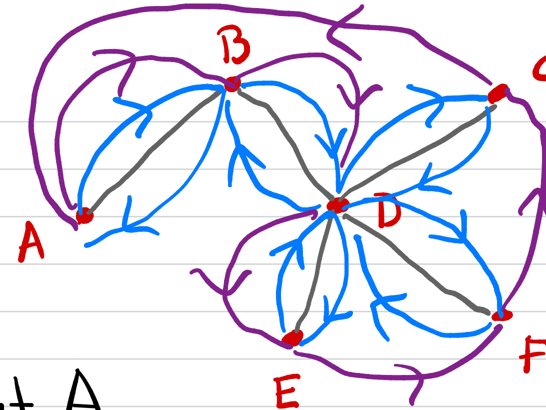


(direct routes are shortest)

Algorithm:

1.) Find the MST T

cost(T) $\leq$ cost(optimal TSP tour)



2.) DFS traversal of T, starting at A

$A \to B \to D \to E \to D \to F \to D \to C \to D \to B \to A$

cost(DFS traversal) = 2 · cost(tree T)

3.) Skip all repeated vertices in the traversal

$A \to B \to D \to E \to F \to C \to A$ = output

cost(TSP tour output) $\leq$ cost(DFS traversal)

$\therefore$ cost(output) $\leq 2 \cdot \mathrm{OPT}$