Two person zero sum games.

- $m \times n$ payoff matrix $A$. 
  - $a_{ij}$ - payoff if row plays $i$ and column plays $j$
- Row mixed strategy: $x = (x_1, \ldots, x_m)$
- Column mixed strategy: $y = (y_1, \ldots, y_n)$
- Payoff for strategy pair $(x,y)$:
  $$p(x,y) = x^tAy$$
- That is,
  $$\sum_{j=1}^{n}(x_jy_j) = \sum_{j=1}^{n}x_j\left(\sum_{i=1}^{m}a_{ij}y_j\right) = \sum_{j=1}^{n}x_ja_{ij}y_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}\right)y_j$$
- Row maximizes, column minimizes
- Equilibrium pair: $(x^*, y^*)$
  $$\left(\begin{array}{c} x^* \\ y^* \end{array}\right)$$
  $$\left(\begin{array}{c} x^* \\ y^* \end{array}\right)'Ay^* = \min_{(x,y)}(x'AY) = \max_{(y,x)}(x'AY)$$
- (No better column strategy, no better row strategy.)

Strategic Games.

- $N$ players.
- Each player has strategy set. $\{S_1, \ldots, S_N\}$
- Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \mathbb{R}^N$)
- Example:
  - 2 players
  - Player 1: $\{\text{Defect, Cooperate}\}$
  - Player 2: $\{\text{Defect, Cooperate}\}$
- Payoff:
  $$\begin{array}{c|cc}
  & C & D \\
  \hline
  C & (3,3) & (0,5) \\
  D & (5,0) & (1,1) \\
  \end{array}$$
- Zero Sum Games. $R = \min_y \max_x (x' Ay)$
- $C = \max_x \min_y (x' Ay)$
- Weak Duality: $R \geq C$.
- Proof: Better to go second. $\square$
  - In situation $R$, $y$ announces "Defense". $x$ plays "Offense."
  - In situation $C$, $x$ announces "Defense". $y$ plays "Offense."
  - Or: if $R > C$, then Column player can play $y_R$ as $y_C$ and do better.
- At Equilibrium $(x^*, y^*)$, payoff $v$:
  - row payoffs $(Ay^*)$ all $\leq v$ $\implies R \leq v$.
  - column payoffs $(x^*A)$ all $\geq v$ $\implies v \leq C$.
  $\implies R \leq v \leq C$
- Equilibrium $\implies R = C!$
- Strong Duality: There is an equilibrium point and $R = C!$
- Doesn't matter who plays first!

Famous because?

- $R P S$
  - $R$}
  - $P$}
  - $S$}
- How do you play?
  - Player 1: play each strategy with equal probability.
  - Player 2: play each strategy with equal probability.
- Definitions.
  - Mixed strategies: Each player plays distribution over strategies.
  - Pure strategies: Each player plays single strategy.
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row).

$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: column knows row cheats.
Why play?
Row is column’s advisor.
... boss.

Equilibrium:

Equilibrium pair: $(x^*, y^*)$?

$p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*$.

(No better column strategy, no better row strategy.)

No row is better:

$\max_i A^{(i)} : y^* = (x^*)^T Ay^*$.  

No column is better:

$\min_y (A^T)^{(i)} : x^* = (x^*)^T Ay^*$.  

$\forall i A^{(i)}$ is $i$th row.

Equilibrium: always?

Equilibrium pair: $(x^*, y^*)$?

$p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*$.  

Does an equilibrium pair: $(x^*, y^*)$, exist?

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes strategy: vector $n$-dimensional vector $y$
... probability distribution over columns.
Payoff $(x, y): x^T Ay$:
nash equilibrium $(x^*, y^*)$:
neither player has better response against others.

If there is an equilibrium: no disadvantage in announcing strategy!
All equilibrium points all have same payoff.

Why? Assume equilibriums: $x^*_1 Ay_1 > x^*_2 Ay_2$,

$\implies \max_i (Ay_i) > \max_i (Ay'_i)$ $x_1$ zero on non-best row of $(Ay_i)$

Best row is worse under $y_2$.

$\implies$ Column player strategy $y_2$ is better than $y_1$
$x_1, y_1$ is not equilibrium. Contradiction.

Equilibrium: play the boss...

Equilibrium Row: $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Column: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
Row Player.

Strategy 1: $\frac{1}{2} \times 0 + \frac{1}{2} \times -1 + \frac{1}{2} \times 1 = -\frac{1}{2}$
Strategy 2: $\frac{1}{2} \times 1 + \frac{1}{2} \times 0 + \frac{1}{2} \times -1 = \frac{1}{2}$
Strategy 3: $\frac{1}{2} \times -1 + \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = 0$
Strategy 4: $\frac{1}{2} \times 0 + \frac{1}{2} \times 0 + \frac{1}{2} \times -1 = -\frac{1}{2}$

Payoff is $0 \times -\frac{1}{2} + \frac{1}{2} \times (\frac{1}{2}) + \frac{1}{2} \times (\frac{1}{2}) + \frac{1}{2} \times (\frac{1}{2}) = \frac{1}{2}$
Column player: every column payoff is $\frac{1}{2}$.
Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!

Zero Sum games and Linear Programs.

Matrix $A$, rows $a_i$, columns, $a^T i$.
Column player maximizes $R$.
Row player minimizes $C$.

$\forall i a^T i \cdot x \geq z$

$\sum_i x_i = 1$

$\forall i a_i \cdot y \leq z$

$\sum y_i = 1$

$y_j \geq 0$

Zero-Sum Games also equivalent to linear programs. Not completely easy.
(Adler, recently.)
**Quest for Polynomial time.**

Dantzig (1947), Kantorovich (1939).

Khachiyan (1979): Ellipsoid method. Impractical...so far.


Best codes are based on simplex, interior point.

Kelner, Spielman: Simplex is polynomial on smooth problems.

Explains performance in practice.

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**Generality of Linear Programming.**

Linear program solves many problems.

How applicable is it?

So applicable that...INSERT JOKE HERE...

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**Circuit Evaluation.**

Circuit Evaluation:
Given: DAG of boolean gates:
Two input AND/OR.
One input NOT.
TRUE/FALSE inputs.

Problem: What is the output of a specified Output Gate?

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**Translation to linear program.**

Variable for gate $g$: $x_g$.

Constraints:

* $0 \leq x_g \leq 1$
* Gate $g$ is true gate: $x_g = 1$.
* Gate $g$ is false gate: $x_g = 0$.

For $\land$ gate:

* $x_g \geq x_h$
* $x_g \geq x_{h'}$
* $x_g \leq x_h + x_{h'}$

For $\lor$ gate:

* $x_g \leq x_h$
* $x_g \leq x_{h'}$
* $x_g \geq x_h + x_{h'} - 1$

For $\neg$ gate:

* $x_g = 1 - x_{h'}$

$x_o$ is 1 if and only if the circuit evaluates to true.

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**61C**

The circuit value problem is completely general!

A computer program can be unfolded into a circuit.

Each level is the circuit for a computer.

The number of levels is the number of steps.

$\iff$ circuit value problems model computation.

$\iff$ linear programs can model any polynomial time problem!

Warning: existence proof, not generally efficient.

Next: NP completeness...more reductions.

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**Lecture in a Minute**

Games

Nash Equilibrium
Zero Sum Two Person Games
Mixed Strategies.
Checking Equilibrium.
Best Response.
Statement of Duality Theorem.

Generality of Linear Program.
Any circuit can be implemented by linear program!
Any polynomial time algorithm
$\Rightarrow$ a poly sized linear program.