Lecture in a Minute

Games
  - Nash Equilibrium
  - Zero Sum Two Person Games
  - Mixed Strategies.
  - Checking Equilibrium.
  - Best Response.
  - Statement of Duality Theorem.

Generality of Linear Program.
  - Any circuit can be implemented by linear program!
  - Any polynomial time algorithm
    $\implies$ a poly sized linear program.
Strategic Games.

\(N\) players.
Each player has strategy set. \(\{S_1, \ldots, S_N\}\).
Vector valued payoff function: \(u(s_1, \ldots, s_n)\) (e.g., \(\in \mathbb{R}^N\)).
Example:

2 players
Player 1: \{ Defect, Cooperate \}.
Player 2: \{ Defect, Cooperate \}.

Payoff:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(3,3)</td>
<td>(0,5)</td>
</tr>
<tr>
<td>D</td>
<td>(5,0)</td>
<td>(1,1)</td>
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Famous because?

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What is the best thing for the players to do?

Both cooperate. Payoff \((3,3)\).

If player 1 wants to do better, what does she do?

Defects! Payoff \((5,0)\)

What does player 2 do now?

Defects! Payoff \((.1,.1)\).

Stable now!

Nash Equilibrium: neither player has incentive to change strategy.
Two person zero sum games.

$m \times n$ payoff matrix $A$.

$a_{i,j}$- payoff if row plays $i$ and column plays $j$

Row mixed strategy: $x = (x_1, \ldots, x_m)$.  
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x, y)$:

$$p(x, y) = x^t Ay$$

That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j} = \sum_i x_i \left( \sum_j a_{i,j} y_j \right) = \sum_i \sum_j x_i a_{i,j} y_j = \sum_j \left( \sum_i x_i a_{i,j} \right) y_j.$$ 

Row maximizes, column minimizes

Equilibrium pair: $(x^*, y^*)$?

$$(x^*)^t Ay^* = \min_y (x^*)^t Ay = \max_x x^t Ay^*.$$ 

(No better column strategy, no better row strategy.)
Zero Sum Games. \[ R = \min_y \max_x (x^t Ay). \]
\[ C = \max_x \min_y (x^t Ay). \]

**Weak Duality:** \( R \geq C. \)

**Proof:** Better to go second.

Note:

- In situation \( R. \) \( y \) announces “Defense”. \( x \) plays “Offense.”
- In situation \( C. \) \( x \) announces “Defense”. \( y \) plays “Offense.”

Or: if \( R > C \), then Column player can play \( y_R \) as \( y_C \) and do better.

At Equilibrium \( (x^*, y^*) \), payoff \( v \):
row payoffs \( (Ay^*) \) all \( \leq v \) \( \implies R \leq v. \)
column payoffs \( ((x^*)^t A) \) all \( \geq v \) \( \implies v \leq C. \)
\( \implies R \leq v \leq C \)

Equilibrium \( \implies R = C! \)

**Strong Duality:** There is an equilibrium point! and \( R = C! \)

Doesn’t matter who plays first!
Roshambo Example.

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<th>P</th>
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<tr>
<td>R</td>
<td>.33</td>
<td>.33</td>
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</tr>
<tr>
<td>P</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

**Mixed strategies:** Each player plays distribution over strategies.

**Pure strategies:** Each player plays single strategy.
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

\[
A = \begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Note: column knows row cheats.
Why play?
Row is column’s advisor.
... boss.
Equilibrium pair: \((x^*, y^*)\)?

\[
p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*.
\]

(No better column strategy, no better row strategy.)

No row is better:

\[
\max_i A^{(i)} \cdot y^* = (x^*)^T Ay^*. \tag{1}
\]

No column is better:

\[
\min_j (A^T)^{(j)} \cdot x^* = (x^*)^T Ay^*.
\]

\(^1 A^{(i)} \) is \(i\)th row.
Equilibrium: play the boss...

\[
A = \begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times 1 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = \frac{1}{6}\)
Strategy 3: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 1 + \frac{1}{6} \times 0 = \frac{1}{6}\)
Strategy 4: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = \frac{1}{6}\)

Payoff is \(0 \times -\frac{1}{3} + \frac{1}{3} \times \left(\frac{1}{6}\right) + \frac{1}{6} \times \left(\frac{1}{6}\right) + \frac{1}{2} \times \left(\frac{1}{6}\right) = \frac{1}{6}\)

Column player: every column payoff is \(\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!
Equilibrium: always?

Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*. \]

Does an equilibrium pair: \((x^*, y^*)\), exist?
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$

row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.

column minimizes strategy: vector $n$-dimensional vector $y$
... probability distribution over columns.

Payoff $(x, y)$: $x^T A y$.

Nash equilibrium $(x^*, y^*)$:
neither player has better response against others.

If there is an equilibrium: no disadvantage in announcing strategy!

All equilibrium points all have same payoff.

Why? Assume equilibriums: $x_1^T A y_1 > x_2^T A y_2$.
\[ \implies \max_i (Ay_1)_i > \max_i (Ay_2)_i \quad x_i \text{ zero on non-best row of } (Ay_1) \]
Best row is worse under $y_2$.
\[ \implies \text{Column player strategy } y_2 \text{ is better than } y_1 \]
$x_1, y_1$ is not equilibrium. Contradiction.
Zero Sum games and Linear Programs.

Matrix $A$, rows $a_i$, columns, $a^{(i)}$.

Row player minimizes $C$.

$$C = \max z$$

$$\forall i \quad a^{(i)} \cdot x \geq z$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Column player maximizes $R$.

$$R = \min z$$

$$\forall i \quad a_i \cdot y \leq z$$

$$\sum_i y_i = 1$$

$$y_i \geq 0$$

Zero-Sum Games also equivalent to linear programs. Not completely easy.

(Adler, recently.)
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
Khachiyan (1979): Ellipsoid method. Impractical so far.

Best codes are based on simplex, interior point. Depends on problem.

Generality of Linear Programming.

Linear program solves many problems.
How applicable is it?
    So applicable that ...INSERT JOKE HERE...
Circuit Evaluation:
Given: DAG of boolean gates:
  two input AND/OR.
  One input NOT.
TRUE/FALSE inputs.
Problem: What is the output of a specified Output Gate?

No really! What is the value of the output?
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$

Gate $g$ is true gate: $x_g = 1$.
Gate $g$ is false gate: $x_g = 0$.

\[ x_g \geq x_h \]
\[ x_g \geq x_{h'} \]
\[ x_g \leq x_h + x_{h'} \]

For $\land$ gate:
\[ x_g \leq x_h, \quad x_g \leq x_{h'} \quad x_g \geq x_h + x_{h'} - 1 \]

For $\neg$ gate: $x_g = 1 - x_h$.

$x_o$ is 1 if and only if the circuit evaluates to true.
The circuit value problem is completely general!
A computer program can be unfolded into a circuit.
Each level is the circuit for a computer.
The number of levels is the number of steps.

⇒ circuit value problems model computation.
⇒ linear programs can model any polynomial time problem!

Warning: existence proof, not generally efficient.

Next: NP completeness..more reductions.
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