Lecture in a Minute

Games
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Games
  Nash Equilibrium
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  Zero Sum Two Person Games
  Mixed Strategies.
  Checking Equilibrium.
  Best Response.
  Statement of Duality Theorem.
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- Nash Equilibrium
- Zero Sum Two Person Games
- Mixed Strategies.
- Checking Equilibrium.
- Best Response.
- Statement of Duality Theorem.

Generality of Linear Program.
Lecture in a Minute

Games
   Nash Equilibrium
   Zero Sum Two Person Games
   Mixed Strategies.
   Checking Equilibrium.
   Best Response.
   Statement of Duality Theorem.

Generality of Linear Program.
   Any circuit can be implemented by linear program!
   Any polynomial time algorithm
      \(\Rightarrow\) a poly sized linear program.
Strategic Games.

$N$ players.
Strategic Games.

$N$ players.
Each player has strategy set. $\{S_1, \ldots, S_N\}$. 
Strategic Games.

$N$ players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \mathbb{R}^N$).
Strategic Games.

$N$ players.
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Example:
Strategic Games.

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Example:
2 players
Strategic Games.

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Example:

2 players
Player 1: $\{\text{Defect, Cooperate}\}$.
Player 2: $\{\text{Defect, Cooperate}\}$. 
Strategic Games.

\(N\) players.
Each player has strategy set. \(\{S_1, \ldots, S_N\}\).
Vector valued payoff function: \(u(s_1, \ldots, s_n)\) (e.g., \(\in \mathbb{R}^N\)).

Example:

2 players

Player 1: \{ Defect, Cooperate \}.
Player 2: \{ Defect, Cooperate \}.

Payoff:
Strategic Games.

$N$ players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

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Player 1: $\{\text{Defect, Cooperate}\}$.
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Payoff:

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Famous because?

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What is the best thing for the players to do?
Famous because?

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What is the best thing for the players to do?

Both cooperate. Payoff \((3,3)\).
What is the best thing for the players to do?

Both cooperate. Payoff (3,3).

If player 1 wants to do better, what does she do?

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Defects! Payoff \((5, 0)\)
What is the best thing for the players to do?
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If player 1 wants to do better, what does she do?
Defects! Payoff $(5,0)$

What does player 2 do now?
Famous because?

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What is the best thing for the players to do?

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What does player 2 do now?
Defects! Payoff (.1,.1).
What is the best thing for the players to do?

Both cooperate. Payoff \((3, 3)\).

If player 1 wants to do better, what does she do?

Defects! Payoff \((5, 0)\)

What does player 2 do now?

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Stable now!
What is the best thing for the players to do?

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Nash Equilibrium:
What is the best thing for the players to do?
Both cooperate. Payoff \((3,3)\).

If player 1 wants to do better, what does she do?
Defects! Payoff \((5,0)\).

What does player 2 do now?
Defects! Payoff \((.1,.1)\).

Stable now!

Nash Equilibrium:
neither player has incentive to change strategy.
Two person zero sum games.

$m \times n$ payoff matrix $A$. 
Two person zero sum games.

$m \times n$ payoff matrix $A$.

$a_{i,j}$- payoff if row plays $i$ and column plays $j$
Two person zero sum games.

$m \times n$ payoff matrix $A$.

- $a_{i,j}$- payoff if row plays $i$ and column plays $j$

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Two person zero sum games.

$m \times n$ payoff matrix $A$.

$a_{i,j}$ - payoff if row plays $i$ and column plays $j$

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$. 

Payoff for strategy pair $(x, y)$:

$p(x, y) = x^t Ay$

That is,

$$
\sum_{i,j} (x_i y_j) a_{i,j} = \sum_{i} x_i \left( \sum_{j} a_{i,j} y_j \right) = \sum_{i} \sum_{j} x_i a_{i,j} y_j = \sum_{j} \left( \sum_{i} x_i a_{i,j} \right) y_j.
$$

Row maximizes, column minimizes

Equilibrium pair: $(x^*, y^*)$ such that

$$
(x^*)^t Ay^* = \min y (x^*)^t Ay = \max x x^t Ay^*.
$$

(No better column strategy, no better row strategy.)
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That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j}$$
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Row maximizes, column minimizes

Equilibrium pair: $(x^*, y^*)$?

$(x^*)^t Ay^* = \min_y (x^*)^t Ay = \max_x x^t Ay$.

(No better column strategy, no better row strategy.)
Two person zero sum games.

$m \times n$ payoff matrix $A$.  

$a_{i,j}$- payoff if row plays $i$ and column plays $j$

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Two person zero sum games.

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Equilibrium pair: $(x^*, y^*)$?

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(No better column strategy,
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Row maximizes, column minimizes

Equilibrium pair: $(x^*, y^*)$?

$$(x^*)^t A y^* = \min_y (x^*)^t A y = \max_x x^t A y^*.$$ 

(No better column strategy, no better row strategy.)
Zero Sum Games. \( R = \min_y \max_x (x^t Ay) \).
Zero Sum Games. 

\[ R = \min_y \max_x (x^t Ay). \]
\[ C = \max_x \min_y (x^t Ay). \]
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**Weak Duality:** \( R \geq C \).

**Proof:** Better to go second.
Zero Sum Games.  \[ R = \min_y \max_x (x^t Ay). \]
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Weak Duality: \( R \geq C. \)

Proof: Better to go second. \( \square \)

Note:
- In situation \( R \). \( y \) announces “Defense”. \( x \) plays “Offense.”
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Zero Sum Games. 

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Or: if \( R > C \), then Column player can play \( y_R \) as \( y_C \) and do better.
Zero Sum Games. \[ R = \min_y \max_x (x^t Ay). \]
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At Equilibrium \((x^*, y^*)\), payoff \( v \):
Zero Sum Games.  
\[ R = \min_y \max_x (x^t Ay). \]
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At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \leq v \)
Zero Sum Games. \[ R = \min_{y} \max_{x} (x^t Ay). \]
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At Equilibrium \( (x^*, y^*) \), payoff \( v \):
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- column payoffs \( ((x^*)^t A) \) all \( \geq v \)
Zero Sum Games. \[ R = \min_y \max_x (x^t Ay). \]
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Zero Sum Games.  

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\( \implies R \leq v \leq C \)
Zero Sum Games. \[ R = \min_y \max_x (x^t Ay). \]
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\( \implies R \leq v \leq C \)

Equilibrium \( \implies R = C! \)
Zero Sum Games. \[ R = \min_y \max_x (x^t Ay). \]
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**Weak Duality:** \( R \geq C. \)

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\( \implies R \leq v \leq C \)

Equilibrium \( \implies R = C! \)

**Strong Duality:** There is an equilibrium point!
Zero Sum Games. \[ R = \min_y \max_x (x^t Ay). \]
\[ C = \max_x \min_y (x^t Ay). \]

Weak Duality: \( R \geq C. \)

Proof: Better to go second.

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At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \leq v \) \( \implies \) \( R \leq v. \)
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\[ \implies R \leq v \leq C \]

Equilibrium \( \implies R = C! \)

Strong Duality: There is an equilibrium point! and \( R = C! \)
Zero Sum Games.  \[ R = \min_y \max_x (x^t Ay). \]
\[ C = \max_x \min_y (x^t Ay). \]

**Weak Duality:** \( R \geq C. \)

**Proof:** Better to go second. \( \square \)

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At Equilibrium \((x^*, y^*)\), payoff \( v \):
- row payoffs \((Ay^*)\) all \( \leq v \) \( \implies R \leq v \).
- column payoffs \(((x^*)^t A)\) all \( \geq v \) \( \implies v \leq C \).

\[ \implies R \leq v \leq C \]

Equilibrium \( \implies R = C! \)

**Strong Duality:** There is an equilibrium point! and \( R = C! \)

Doesn’t matter who plays first!
Roshambo Example.

How do you play?

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Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.
Roshambo Example.

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How do you play?

Player 1: play each strategy with equal probability.
### Roshambo Example.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.
Roshambo Example.

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Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.
Roshambo Example.

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How do you play?

Player 1: play each strategy with equal probability.
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Definitions.

**Mixed strategies:** Each player plays distribution over strategies.
Roshambo Example.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

*Mixed strategies:* Each player plays distribution over strategies.

*Pure strategies:* Each player plays single strategy.
Playing the boss...

Row has extra strategy: Cheat.
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:

Rock is strategy 1, Paper is 2, Scissors is 3,
and Cheat is 4 (for row.)

\[
\begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Note: column knows row cheats.

Why play?
Row is column's advisor.
... boss.
Playing the boss...

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Why play?
Row is column’s advisor.
... boss.
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^T A y^* = \max_y (x^*)^T A y \]

No better row strategy, no better column strategy.

\[ \max_i A(i) \cdot y^* = (x^*)^T A y^* \]

No column is better:

\[ \min_j (A^T)(j) \cdot x^* = (x^*)^T A y^* \]

\(^1 A^{(i)}\) is \(i\)th row.
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[
p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*.
\]

(No better column strategy, no better row strategy.)

\(^1\)\(A^{(i)}\) is \(i\)th row.
Equilibrium.

Equilibrium pair: $(x^*, y^*)$?

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(No better column strategy, no better row strategy.)

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\[ \max_i A^{(i)} \cdot y^* = (x^*)^T Ay^*. \]

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Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*. \]

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No row is better:
\[ \max_i A^{(i)} \cdot y^* = (x^*)^T Ay^*. \]

No column is better:
\[ \min_j (A^T)^{(j)} \cdot x^* = (x^*)^T Ay^*. \]

\(^1 A^{(i)} \) is \( i \)th row.
Equilibrium: play the boss...

\[ A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Equilibrium:

Row:
\( (0, 1, 3, 1, 6, 1, 2) \).

Column:
\( (1, 3, 1, 2, 1, 6) \).

Payoff?
Remember: weighted average of pure strategies.

Row Player.
Strategy 1:
\[ 1\frac{1}{3} \times 0 + 1\frac{1}{2} \times (1 - 1) + 1\frac{5}{6} \times 1 = 1 \frac{1}{3} \]

Strategy 2:
\[ 1\frac{1}{3} \times 1 + 1\frac{1}{2} \times 0 + 1\frac{5}{6} \times (1 - 1) = 1 \frac{1}{6} \]

Strategy 3:
\[ 1\frac{1}{3} \times (1 - 1) + 1\frac{1}{2} \times 1 + 1\frac{5}{6} \times 0 = 1 \frac{1}{6} \]

Strategy 4:
\[ 1\frac{1}{3} \times 0 + 1\frac{1}{2} \times 0 + 1\frac{5}{6} \times (1 - 1) = 1 \frac{1}{6} \]

Payoff is:
\[ 0 \times 1\frac{1}{3} + 1\frac{1}{3} \times (1\frac{1}{6}) + 1\frac{1}{2} \times (1\frac{1}{6}) + 1\frac{5}{6} \times (1\frac{1}{6}) = 1\frac{1}{6} \]

Column player: every column payoff is \( 1\frac{1}{6} \).

Both only play optimal strategies!

Complementary slackness.

Why play more than one?
Limit opponent payoff!
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\).
Equilibrium: play the boss...

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A = \begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
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Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).
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Payoff?
Equilibrium: play the boss...

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Equilibrium: play the boss...

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Row Player.
Equilibrium: play the boss...

A =

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Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1\)
Equilibrium: play the boss...

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Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


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Equilibrium: Row: $\left(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$. Column: $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$.


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Payoff is $0 \times -\frac{1}{3} + \frac{1}{3} \times \left(\frac{1}{6}\right) + \frac{1}{6} \times \left(\frac{1}{6}\right) + \frac{1}{2} \times \left(\frac{1}{6}\right)$
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Column player: every column payoff is $\frac{1}{6}$. 
Equilibrium: play the boss...

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A = \begin{bmatrix}
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\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}\)
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Column player: every column payoff is \(\frac{1}{6}\).

Both only play optimal strategies!
Equilibrium: play the boss...

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Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


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Column player: every column payoff is \(\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness.
Equilibrium: play the boss...

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\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times 1 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = \frac{1}{6}\)
Strategy 3: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 1 + \frac{1}{6} \times 0 = \frac{1}{6}\)
Strategy 4: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = \frac{1}{6}\)

Payoff is \(0 \times -\frac{1}{3} + \frac{1}{3} \times (\frac{1}{6}) + \frac{1}{6} \times (\frac{1}{6}) + \frac{1}{2} \times (\frac{1}{6}) = \frac{1}{6}\)

Column player: every column payoff is \(\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness.

Why play more than one?
Equilibrium: play the boss...

\[
A = \begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times 1 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = \frac{1}{6}\)
Strategy 3: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 1 + \frac{1}{6} \times 0 = \frac{1}{6}\)
Strategy 4: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = \frac{1}{6}\)

Payoff is \(0 \times -\frac{1}{3} + \frac{1}{3} \times (\frac{1}{6}) + \frac{1}{6} \times (\frac{1}{6}) + \frac{1}{2} \times (\frac{1}{6}) = \frac{1}{6}\)

Column player: every column payoff is \(\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness.

Why play more than one? Limit opponent payoff!
Equilibrium: always?

Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x x^T Ay^*. \]
Equilibrium: always?

Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^T A y^* = \min_y (x^*)^T A y = \max_x x^T A y^*. \]

Does an equilibrium pair: \((x^*, y^*)\), exist?
Equilibrium value is unique.

Zero sum game:

\[
\begin{align*}
\text{Row maximizes strategy:} & \quad \text{m-dimensional vector } x, \\
\text{Column minimizes strategy:} & \quad \text{n-dimensional vector } y,
\end{align*}
\]

Payoff \((x, y)\):

\[
x^T Ay.
\]

Nash equilibrium \((x^*, y^*)\):

Neither player has better response against others.

If there is an equilibrium: no disadvantage in announcing strategy!

All equilibrium points all have same payoff.

Why? Assume equilibriums:

\[
x^T_1 Ay_1 > x^T_2 Ay_2.
\]

\[
\Rightarrow \max_i (Ay_1)_i > \max_i (Ay_2)_i.
\]

Best row is worse under \(y_2\).

\[
\Rightarrow \text{Column player strategy } y_2 \text{ is better than } y_1, x_1, y_1 \text{ is not equilibrium.}
\]

Contradiction.
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes.

Payoff $(x, y)$:
$x^T A y$.

Nash equilibrium $(x^*, y^*)$:
neither player has better response against others.
If there is an equilibrium: no disadvantage in announcing strategy!
All equilibrium points all have same payoff.
Why? Assume equilibriums:
$x^T_1 A y_1 > x^T_2 A y_2$.
$\Rightarrow$ \( \max_i (A y_1)_i > \max_i (A y_2)_i \)
x_i zero on non-best row of $(A y_1)$
Best row is worse under $y_2$.
$\Rightarrow$ Column player strategy $y_2$ is better than $y_1$
x_1, y_1 is not equilibrium.
Contradiction.
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes. strategy: vector $n$-dimensional vector $y$
... probability distribution over columns.
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes strategy: vector $n$-dimensional vector $y$
... probability distribution over columns.
Payoff $(x, y)$: $x^TAy$. 
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes. strategy: vector $n$-dimensional vector $y$
... probability distribution over columns.
Payoff $(x, y)$: $x^T A y$.
Nash equilibrium $(x^*, y^*)$: 
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes. strategy: vector $n$-dimensional vector $y$
... probability distribution over columns.

Payoff $(x, y)$: $x^T Ay$.

Nash equilibrium $(x^*, y^*)$:
neither player has better response against others.
Equilibrium value is unique.

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All equilibrium points all have same payoff.
Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$. 
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
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Payoff \((x, y)\): \( x^T A y \).

Nash equilibrium \((x^*, y^*)\):
neither player has better response against others.

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All equilibrium points all have same payoff.

Why? Assume equilibriums: \( x_1^T A y_1 > x_2^T A y_2 \).
\[ \implies \max_i (A y_1)_i > \max_i (A y_2)_i \]
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Zero sum game: $m \times n$ matrix $A$
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Payoff $(x, y)$: $x^T Ay$.

Nash equilibrium $(x^*, y^*)$:
neither player has better response against others.

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All equilibrium points all have same payoff.

Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$.
$\implies \max_i (Ay_1)_i > \max_i (Ay_2)_i$; $x_i$ zero on non-best row of $(Ay_1)$
Equilibrium value is unique.

Zero sum game: \( m \times n \) matrix \( A \)
row maximizes strategy: \( m \)-dimensional vector \( x \)
... probability distribution over rows.
column minimizes. strategy: vector \( n \)-dimensional vector \( y \)
... probability distribution over columns.

Payoff \( (x, y) \): \( x^T Ay \).

Nash equilibrium \( (x^*, y^*) \):
neither player has better response against others.

If there is an equilibrium: no disadvantage in announcing strategy!

All equilibrium points all have same payoff.

Why? Assume equilibriums: \( x_1^T Ay_1 > x_2^T Ay_2 \).
\[ \implies \max_i (Ay_1)_i > \max_i (Ay_2)_i \quad x_i \text{ zero on non-best row of } (Ay_1) \]

Best row is worse under \( y_2 \).
Equilibrium value is unique.

Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.
column minimizes. strategy: vector $n$-dimensional vector $y$
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Payoff $(x, y)$: $x^T Ay$.
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All equilibrium points all have same payoff.
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row maximizes strategy: \( m \)-dimensional vector \( x \)
... probability distribution over rows.

column minimizes. strategy: vector \( n \)-dimensional vector \( y \)
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Payoff \((x, y)\): \( x^T A y \).

Nash equilibrium \((x^*, y^*)\):
neither player has better response against others.

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All equilibrium points all have same payoff.

Why? Assume equilibriums: \( x_1^T A y_1 > x_2^T A y_2 \).
\[ \implies \max_i (Ay_1)_i > \max_i (Ay_2)_i, \quad x_i \text{ zero on non-best row of } (Ay_1) \]
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$x_1, y_1$ is not equilibrium.
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Zero sum game: $m \times n$ matrix $A$
row maximizes strategy: $m$-dimensional vector $x$
... probability distribution over rows.

column minimizes strategy: vector $n$-dimensional vector $y$
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Payoff $(x, y)$: $x^T Ay$.

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Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$.

$\implies \max_i(Ay_1)_i > \max_i(Ay_2)_i$, $x_i$ zero on non-best row of $(Ay_1)$
Best row is worse under $y_2$.

$\implies$ Column player strategy $y_2$ is better than $y_1$

$x_1, y_1$ is not equilibrium. Contradiction.
Zero Sum games and Linear Programs.

Matrix $A$, rows $a_i$, columns, $a^{(i)}$. 

Row player minimizes $C$.

$$C = \max_z \forall i \ a(i) \cdot x \geq z \sum_i x \geq 1 \quad x \geq 0$$

Column player maximizes $R$.

$$R = \min_z \forall i \ a_i \cdot y \leq z \sum_i y \geq 1 \quad y \geq 0$$

Zero-Sum Games also equivalent to linear programs. Not completely easy. (Adler, recently.)
Zero Sum games and Linear Programs.

Matrix $A$, rows $a_i$, columns, $a^{(i)}$.

Row player minimizes $C$.

$$C = \max z$$

$$\forall i \quad a^{(i)} \cdot x \geq z$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$
Zero Sum games and Linear Programs.

Matrix $A$, rows $a_i$, columns, $a^{(i)}$.

Row player minimizes $C$.

$$\begin{align*}
C &= \max z \\
\forall i \quad a^{(i)} \cdot x \geq z \\
\sum_i x_i &= 1 \\
x_i &\geq 0
\end{align*}$$

Column player maximizes $R$.

$$\begin{align*}
R &= \min z \\
\forall i \quad a_i \cdot y \leq z \\
\sum_i y_i &= 1 \\
y_i &\geq 0
\end{align*}$$

Zero-Sum Games also equivalent to linear programs. Not completely easy. (Adler, recently.)
Zero Sum games and Linear Programs.

Matrix $A$, rows $a_i$, columns, $a^{(i)}$.

Row player minimizes $C$.

$$C = \max z$$

$$\forall i \quad a^{(i)} \cdot x \geq z$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Column player maximizes $R$.

$$R = \min z$$

$$\forall i \quad a_i \cdot y \leq z$$

$$\sum_i y_i = 1$$

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Zero-Sum Games also equivalent to linear programs. Not completely easy.
Zero Sum games and Linear Programs.

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$$R = \min z$$

$$\forall i \quad a_i \cdot y \leq z$$

$$\sum_i y_i = 1$$

$$y_i \geq 0$$

Zero-Sum Games also equivalent to linear programs. Not completely easy.

(Adler, recently.)
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
Quest for Polynomial time.

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Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
Khachiyan (1979): Ellipsoid method. Impractical..
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).

Khachiyan (1979): Ellipsoid method. Impractical ..so far.
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
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Karmarkar (at Berkeley): Interior point methods.
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
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Best codes are based on simplex, interior point.
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
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Best codes are based on simplex, interior point. Depends on problem.
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
Khachiyan (1979): Ellipsoid method. Impractical ..so far.
Karmarkar (at Berkeley): Interior point methods.
Fast in practice.
Best codes are based on simplex, interior point.
    Depends on problem.
Kelner,Spielman: Simplex is polynomial on smooth problems.
Quest for Polynomial time.

Dantzig (1947), Kantorovich (1939).
Khachiyan (1979): Ellipsoid method. Impractical ..so far.

Best codes are based on simplex, interior point. Depends on problem.

Generality of Linear Programming.

Linear program solves many problems.
Generality of Linear Programming.

Linear program solves many problems.
How applicable is it?
Generality of Linear Programming.

Linear program solves many problems.
How applicable is it?
    So applicable that ...
Generality of Linear Programming.

Linear program solves many problems.
How applicable is it?
   So applicable that ...INSERT JOKE HERE...
Circuit Evaluation.

Circuit Evaluation:

Given: DAG of boolean gates: two input AND/OR, one input NOT. TRUE/FALSE inputs.

Problem: What is the output of a specified Output Gate?

No really! What is the value of the output?
Circuit Evaluation:
Given: DAG of boolean gates:
Circuit Evaluation.

Circuit Evaluation:
Given: DAG of boolean gates:
   two input AND/OR.
Circuit Evaluation:
Given: DAG of boolean gates:
   two input AND/OR.
   One input NOT.

No really!
What is the value of the output?
Circuit Evaluation:
Given: DAG of boolean gates:
   two input **AND/OR**.
   One input **NOT**.
**TRUE/FALSE** inputs.

No really! What is the value of the output?
Circuit Evaluation:
Given: DAG of boolean gates:
  two input AND/OR.
  One input NOT.
TRUE/FALSE inputs.

Problem: What is the output of a specified Output Gate?
Circuit Evaluation:
Given: DAG of boolean gates:
- Two input \textbf{AND/OR}.
- One input \textbf{NOT}.
- \textbf{TRUE/FALSE} inputs.

Problem: What is the output of a specified Output Gate?

\[ \text{output} \]

\[ \wedge \]

\[ \neg \]

\[ \vee \]

\[ \text{output} \]

\[ T \]

\[ F \]

\[ T \]
Circuit Evaluation:
Given: DAG of boolean gates:

two input **AND/OR**.

One input **NOT**.

**TRUE/FALSE** inputs.

Problem: What is the output of a specified Output Gate?

No really!
Circuit Evaluation:
Given: DAG of boolean gates:
  two input \textbf{AND/OR}.
  One input \textbf{NOT}.
\textbf{TRUE/FALSE} inputs.

Problem: What is the output of a specified Output Gate?

No really! What is the value of the output?
Translation to linear program.

Variable for gate $g$: $x_g$. 

Gate $g$ is true gate: $x_g = 1$.

Gate $g$ is false gate: $x_g = 0$.

\[ x_g \geq x_h \]
\[ x_g \geq x_{h'} \]
\[ x_g \leq x_h + x_{h'} - 1 \]

For $\land$ gate:
\[ x_g \leq x_h, \quad x_g \leq x_{h'} \]
\[ x_g \geq x_h + x_{h'} - 1 \]

For $\neg$ gate:
\[ x_g = 1 - x_h. \]
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$
Variable for gate $g$: $x_g$.

Constraints:

$0 \leq x_g \leq 1$

Gate $g$ is true gate: $x_g = 1$. 

Gate $g$ is false gate: $x_g = 0$. 

$\lor \quad h \quad h' \quad x_g \geq x_h \quad x_g \geq x_h' \quad x_g \leq x_h + x_h' - 1$

For $\land$ gate:

$x_g \leq x_h, \quad x_g \leq x_h' \quad x_g \geq x_h + x_h' - 1$

For $\neg$ gate:

$x_g = 1 - x_h$. 

$x_o$ is 1 if and only if the circuit evaluates to true.
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$
Gate $g$ is true gate: $x_g = 1$.
Gate $g$ is false gate: $x_g = 0$. 
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
0 $\leq x_g \leq 1$
Gate $g$ is true gate: $x_g = 1$.
Gate $g$ is false gate: $x_g = 0$. 

\begin{align*}
0 &\leq x_g &\leq 1 \\
\land \text{ gate: } &x_g \leq x_h \land x_g \leq x_{h'} \\
\lor \text{ gate: } &x_g \leq x_h + x_{h'} - 1 \\
\lnot \text{ gate: } &x_g = 1 - x_h
\end{align*}
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$
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Translation to linear program.

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$\bigvee h' \geq x_h$

$h'$

$\bigvee$

$h$
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
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$$x_g \geq x_h$$
$$x_g \geq x_{h'}$$
Variable for gate \( g \): \( x_g \).

Constraints:
\( 0 \leq x_g \leq 1 \)
Gate \( g \) is true gate: \( x_g = 1 \).
Gate \( g \) is false gate: \( x_g = 0 \).

\[
\begin{align*}
    x_g &\geq x_h \\
    x_g &\geq x_{h'} \\
    x_g &\leq x_h + x_{h'}
\end{align*}
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\[
x_g \geq x_h \\
\geq x_{h'} \#
\]
\[
x_g \leq x_h + x_{h'} \\
\]

For $\land$ gate:
\[
x_g \leq x_h, \quad x_g \leq x_h' \\
\]
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$
Gate $g$ is true gate: $x_g = 1$.
Gate $g$ is false gate: $x_g = 0$.

For $\lor$ gate:
$x_g \geq x_h$
$x_g \geq x_{h'}$
$x_g \leq x_h + x_{h'}$

For $\land$ gate:
$x_g \leq x_h$, $x_g \leq x'_{h}$
$x_g \geq x_h + x_{h'} - 1$
Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$
Gate $g$ is true gate: $x_g = 1$.
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\[ x_g = 1 - x_h. \]
Translation to linear program.

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Gate $g$ is false gate: $x_g = 0$.

\[
\begin{align*}
  x_g & \geq x_h \\
  x_g & \geq x_{h'} \\
  x_g & \leq x_h + x_{h'}
\end{align*}
\]

For $\land$ gate:
\[x_g \leq x_h, \quad x_g \leq x_{h'}, \quad x_g \geq x_h + x_{h'} - 1\]

For $\neg$ gate: $x_g = 1 - x_h$.

$x_o$ is 1 if and only if the circuit evaluates to true.
The circuit value problem is completely general!
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A computer program can be unfolded into a circuit.
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Each level is the circuit for a computer.
The circuit value problem is completely general! A computer program can be unfolded into a circuit. Each level is the circuit for a computer. The number of levels is the number of steps.
The circuit value problem is completely general!
A computer program can be unfolded into a circuit.
Each level is the circuit for a computer.
The number of levels is the number of steps.

⇒ circuit value problems model computation.
The circuit value problem is completely general!
A computer program can be unfolded into a circuit.
Each level is the circuit for a computer.
The number of levels is the number of steps.
⇒ circuit value problems model computation.
⇒⇒ linear programs can model any polynomial time problem!
The circuit value problem is completely general!
A computer program can be unfolded into a circuit.
Each level is the circuit for a computer.
The number of levels is the number of steps.
\[\implies\text{circuit value problems model computation.}\]
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Warning: existence proof, not generally efficient.
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⇒ circuit value problems model computation.
⇒ linear programs can model any polynomial time problem!
Warning: existence proof, not generally efficient.
Next: NP completeness..more reductions.
Lecture in a Minute

Games
Lecture in a Minute

Games
  Nash Equilibrium
Lecture in a Minute

Games
  Nash Equilibrium
  Zero Sum Two Person Games
  Mixed Strategies.
  Checking Equilibrium.
  Best Response.
  Statement of Duality Theorem.
Lecture in a Minute

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Generality of Linear Program.
Lecture in a Minute

Games
  Nash Equilibrium
  Zero Sum Two Person Games
  Mixed Strategies.
  Checking Equilibrium.
  Best Response.
  Statement of Duality Theorem.

Generality of Linear Program.
  Any circuit can be implemented by linear program!
  Any polynomial time algorithm
    $\implies$ a poly sized linear program.