Application of Multiplicative Weights.

Complementary slackness.
Remember: weighted average of pure strategies.

Two person zero sum games.

Lecture in a minute

Multiplicative Weights
⇒ strong duality for Zero-Sum Games.

Row player evolves MW distribution.
Row player plays best response.
Output average of column player as $y$.
Output average of row player as $x$.

Equilibrium: play the boss...
Both only play optimal strategies!

Equilibrium.
Equilibrium pair: $(x^*,y^*)$?

Equilibrium: Row: $(0, \frac{1}{6}, \frac{1}{6}, ...)$ is $\frac{1}{6}$.

Column player plays MW distribution.

Equilibrium: play the boss...

Equilibrium: Row: $(0, \frac{1}{6}, \frac{1}{6}, ...)$ is $\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!

Equilibrium: (0, $\frac{1}{6}$, $\frac{1}{6}$). Column: ($\frac{1}{6}$, $\frac{1}{6}$).

Payoff? Without learning, weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{2} \times 0 + \frac{1}{2} \times -1 + \frac{1}{2} \times 1 = -\frac{1}{2}$
Strategy 2: $\frac{1}{2} \times 0 + \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}$
Strategy 3: $\frac{1}{2} \times 0 + \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = 0$
Strategy 4: $\frac{1}{2} \times 0 + \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}$

Payoff is $0 \times -\frac{1}{2} + \frac{1}{2} \times \left(\frac{1}{2}\right) + \frac{1}{2} \times \left(\frac{1}{2}\right) + \frac{1}{2} \times \left(\frac{1}{2}\right) = \frac{1}{2}$

Column player: every column payoff is $\frac{1}{2}$.

Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!

Matrix Reminders.

$m \times n$ matrix $A$.
$m$-dimensional vector $x$.
$n$-dimensional vector $y$.

$A^T$ is $n$-dimensional (column) vector.

Equilibrium: strong duality for Zero-Sum Games.

Recall row maximizes, column minimizes.

BTW: Only the circles mean anything.

Strong duality for Zero-Sum Games.

$x^T A$ is $m$-dimensional (column) vector.

Equilibrium: weak duality for Zero-Sum Games.

Weak duality for Zero-Sum Games.

$x^T A$ is $m$-dimensional (column) vector.

Equilibrium: strong duality for Zero-Sum Games.

Weak duality for Zero-Sum Games.

$x^T A$ is $m$-dimensional (column) vector.

Equilibrium: weak duality for Zero-Sum Games.

Weak duality for Zero-Sum Games.

$x^T A$ is $m$-dimensional (column) vector.
Equilibrium: always?

Equilibrium pair: $(x^*, y^*)$?

$\rho(x, y) = (x^*)^T A y^* - \min_{x} (x^*)^T A y = \max_{y} (x^*)^T A y^*$.

Does an equilibrium pair: $(x^*, y^*)$, exist?

Proof of Equilibrium.

Sort of.

Approximate equilibrium ...

$C(x) = \min_{y} x^T A y$

$R(y) = \max_{x} x^T A y$

Strategy pair: $(x, y)$

Equilibrium: $(x, y)$

$R(y) = C(x) \implies R(y) - C(x) = 0$.

Approximate Equilibrium: $R(y) - C(x) \leq \varepsilon$.

With $R(y) > C(x)$ (weak duality)

$\rightarrow$ “Response $y$ to $x$ is within $\varepsilon$ of best response”

$\rightarrow$ “Response $x$ to $y$ is within $\varepsilon$ of best response”

Best Response

Column goes first:

Find $y$, where best row is not too high.

$$R = \min_{y} \max_{x} (x^T A y).$$

Note: $x$ can be $(0, 0, 1, 0)$.

Example: Roshambo. Value of $R$?

Row goes first:

Find $x$, where best column is not low.

$$C = \max_{y} \min_{x} (x^T A y).$$

Agin: $y$ of form $(0, 0, 0, 0)$. Example: Roshambo. Value of $C$?

Proof of approximate equilibrium.

How?

(A) Using geometry.

(B) Using a fixed point theorem.

(C) Using multiplicative weights.

(C)

Not hard. Even easy. Still, head scratching happens.

Duality.

$$R = \min_{x} \max_{y} (x^T A y).$$

$$C = \max_{y} \min_{x} (x^T A y).$$

Weak Duality: $R \geq C$.

Proof: Better to go second.

At Equilibrium $(x^*, y^*)$, payoff $v$: row payoffs $(A y^*)$ all $\leq v \implies R \leq v$.

column payoffs $(x^*)^T A$ all $\geq v \implies v \geq C$.

$\implies R \leq C$

Equilibrium $\implies R = C!$

Strong Duality: There is an equilibrium point! and $R = C!$

Doesn’t matter who plays first!

Games and experts

Again: find $(x^*, y^*)$, such that

$$(\max_{x} x A y^*) - (\min_{y} x^T A y) \leq \varepsilon$$

$$R(x^*) - C(y^*) \leq \varepsilon$$

Experts Framework:

$n$ Experts. $T$ days. $L^*$ -total loss of best expert.

Multiplicative Weights Method yields loss $L$ where

$L \leq (1 + \varepsilon) L^* + \frac{n \varepsilon }{T}$
Games and Experts.

Assume: A has payoffs in [0,1].
For $T = \frac{\ln n}{\epsilon^2}$ days:
1) $m$ pure column strategies are experts.
Use multiplicative weights, produce column distribution.
Let $y_t$ be distribution (column strategy) on day $t$.
2) Each day, adversary plays best row response to $y_t$.
Choose row of $A$ that maximizes column’s expected loss.
Let $x_t$ be indicator vector for this row.

Wrapping up duality theorem.

For any $\epsilon$, there exists an $\epsilon$-Approximate Equilibrium.
Does an equilibrium exist? Yes.
Something about math here? Fixed point theorem.
Other proofs: use geometry, linear programming.

Complexity?
$T = \frac{\ln n}{\epsilon^2} \rightarrow O(n \ln^2 2)$. Basically linear!
Versus Linear Programming: $O(n^2 m)$ Basically quadratic.
(Faster linear programming: $O(\sqrt{nm})$ linear solution solves.)
Still much slower ... and more complicated.
Also works with both using multiplicative weights.
“In practice.”

Approximate Equilibrium!

Experts: $y_t$ is MW strategy on day $t$, $x_t$ is best row against $y_t$.
Let $x^* = \frac{1}{T} \sum x_t$ and $y^* = \arg\min_y x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$ for $T = \frac{\ln n}{\epsilon^2}$.

Row payoff: $R(y^*) = \max_x x A y^*$.
Loss on day $t$, $x_t A y_t \geq R(y^*)$ by the choice of $y^*$.
Thus, algorithm loss, $L$, is $\geq T \times R(y^*)$.

Best expert: $L^*$ - best column against the row distributions played.

Multiplicative Weights: $L \leq (1+\epsilon) L^* + \frac{\ln n}{\epsilon^2}$
$T \times R(y^*) \leq (1+\epsilon) T \times C(x^*) + \frac{\ln n}{\epsilon^2}$
$\rightarrow R(y^*) - C(x^*) \leq \epsilon C(y^*) + \frac{\ln n}{\epsilon^2}$
$T = \frac{\ln n}{\epsilon^2}$, $C(x^*) \leq 1$
$\rightarrow R(y^*) - C(x^*) \leq 2\epsilon$.

Boosting.

This is for fun. Not testable.

Boosting...

Get labelled dataset.
Learn model 1.
Take points previous models don’t learn well.
Learn model 2.
Repeat.
Combine models $1 \ldots n$, for better predictor?
How? Majority Vote.
How to analyse?
Boosting Example.

Learning just a bit.
Example: set of labelled points, find hyperplane that separates.

Boosting/MW Framework

Experts are points. “Adversary” is weak learner.

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.
Can we do this?
(A) Yes
(B) No
If yes. How?
Multiplicative Weights!
The endpoint to a line of research.

Calculation..

|Sbad|(1−ε)T/2 ≤ ne−ε(1/2+γ)T
Set ε = γ, take logs.
ln (|Sbad|) + 1/2 ln(1−γ) ≤ −γT(1/2 + γ)
Again, −γ−γ2 ≤ ln(1−γ),
ln (|Sbad|) + 1/2 (−γ−γ2) ≤ −γT(1/2 + γ) → ln (|Sbad|) ≤ −γ2T

And T = 2 ln µ,
ln (|Sbad|) ≤ ln µ − |Sbad| ≤ µ.
The misclassified set is at most µ fraction of all the points.
The hypothesis correctly classifies 1 − µ fraction of the points !

Claim: Multiplicative weights: h(x) is correct on 1 − µ of the points !
Some details...

Weak learner learns over distributions of points not points.
Make copies of points to simulate distributions.

Conclusion.

Standard method in practice for machine learning for combining repeated base learning algorithms.