# CS 170
# Efficient Algorithms and Intractable Problems

## Lecture 2:
### Divide and Conquer I, Asymptotics

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# Announcements

1. OH schedule is finalized and on the course calendar!
   - Some rooms may change, keep checking the calendar for the location!
2. Discussion schedule finalized tomorrow. Check discussion tab on webpage. Start discussions next week.
3. HW1 will be realized this Sunday, stay tuned
   - If you aren't on Gradescope, send private post on Edstem.
4. Lecture recordings: Maybe released 24 hours later due to post-processing needed.

# Recap of last time

Introductions all around!

Our motivating questions about algorithms:

- Does it work?
- Is it fast?
- Can I do better?

Technical content:

- Arithmetic and Big Oh notation
- Intro to Divide and Conquer
- First attempt at fast multiplication

→ Still didn't beat $O(n^2)$

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## The algorithm

(simplify: assume even $n$)

Break up the multiplication of two integers with $n$ digits into multiplication of integers with $n/2$ digits:

$$[x_1 x_2 \cdots x_n] = \overbrace{[x_1, x_2, \cdots, x_{n/2}]}^{a} \times 10^{\frac{n}{2}} + \overbrace{[x_{n/2+1} x_{n/2+2} \cdots x_n]}^{b}$$

$$x \times y = \left(a \times 10^{\frac{n}{2}} + b\right)\left(c \times 10^{\frac{n}{2}} + d\right)$$
$$= \underbrace{(a \times c)}_{P1} 10^{n} + (\underbrace{a \times d}_{P2} + \underbrace{c \times b}_{P3}) 10^{n/2} + \underbrace{(b \times d)}_{P4}$$

One $n$-digit multiplication → Four $n/2$-digit multiplications

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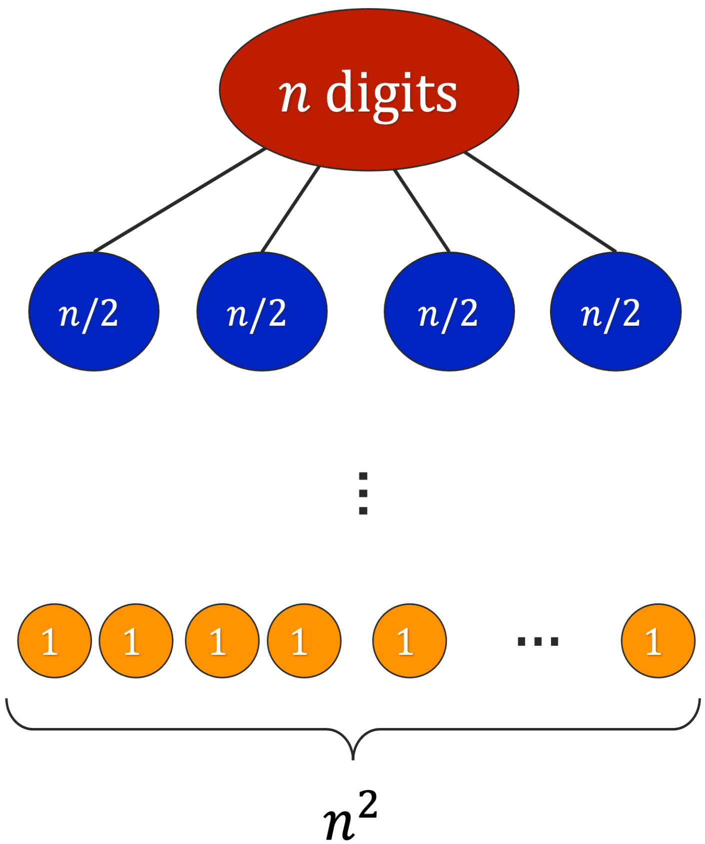



| Layer | # of digits | # problems |
|---|---|---|
| 0 | $n$ | 1 |
| 1 | $n/2$ | 4 |
| ⋮ | ⋮ | ⋮ |
| $t$ | $\frac{n}{2^t}$ | $4^t$ |
| ⋮ | ⋮ | ⋮ |
| $\log_2(n)$ | 1 | $4^{\log_2 n} = n^2$ |

# This lecture

- Karatsuba's algorithm with $O(n^{1.6})$

→Using divide and conquer, but this time better!

- Reviewing $O(\cdot)$ and $\Omega(\cdot)$ notation formally.

- Recurrence relations and a useful theorem for solving them!

# Karatsuba's Idea

Divide and Conquer indeed can lead to a faster algorithm!

$$x \times y = \left(a \times 10^{\frac{n}{2}} + b\right)\left(c \times 10^{\frac{n}{2}} + d\right)$$

$$= \underbrace{(a \times c)}_{P1} 10^{n} + (\underbrace{a \times d}_{P2} + \underbrace{c \times b}_{P3}) 10^{n/2} + \underbrace{(b \times d)}_{P4}$$

The issue is that we are creating 4 sub-problems.
What if we could create fewer subproblems?

**Main idea:** Could we write P2+P3 using what we compute in P1 and P4, and at most one other $n/2$-digit multiplication?

Well… Gauss had used this trick too, for complex numbers…

# Karatsuba's Clever Trick

Let us only compute 3 multiplications with $n/2$ digit numbers:

- Q1: $a \times c$
- Q2: $b \times d$
- Q3: $(a+b)(c+d)$

**Expressing P2+P3 differently**

$$ad + cb = (a+b)(c+d) - \mathrm{ac} - \mathrm{bd}$$

**Three subproblems**

$$x \times y = \left(a \times 10^{\frac{n}{2}} + b\right)\left(c \times 10^{\frac{n}{2}} + d\right)$$

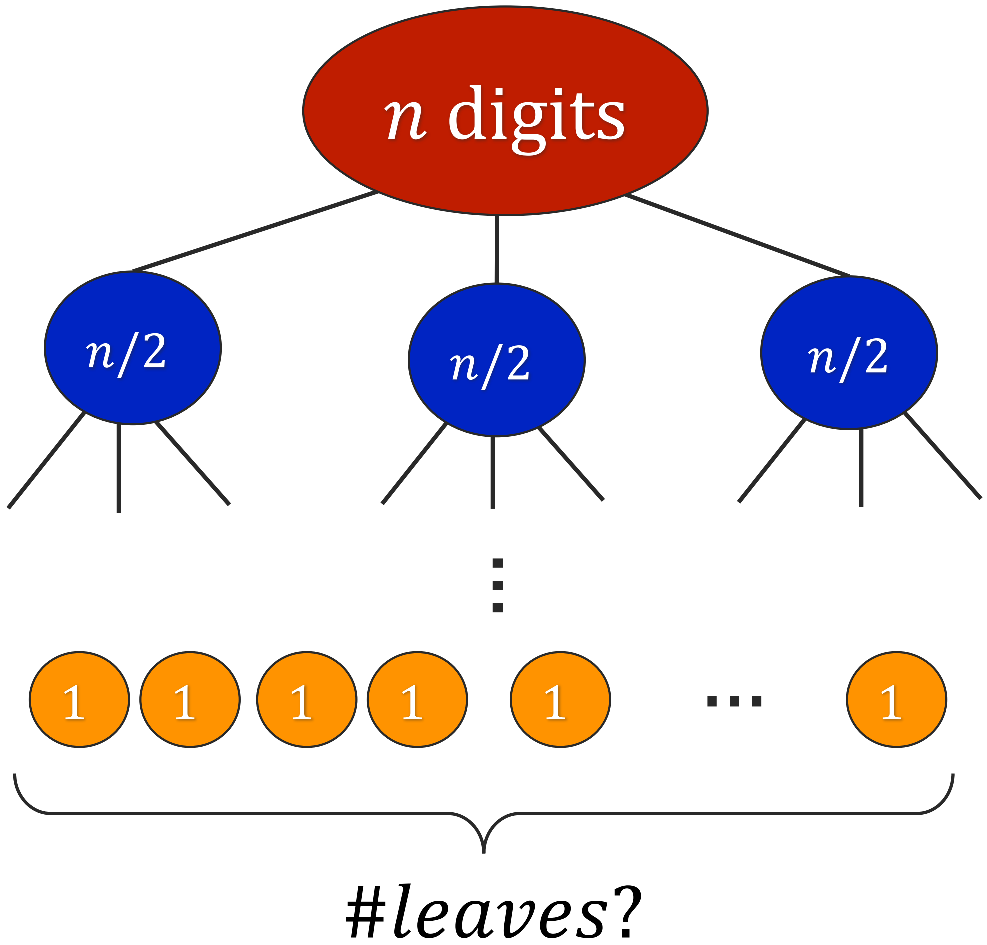
$$= \underbrace{(a \times c)}_{\text{Q1}} 10^{n} + \underbrace{(a \times d + c \times b)}_{\text{Q3}-\text{Q1}-\text{Q2}} 10^{n/2} + \underbrace{(b \times d)}_{\text{Q2}}$$

What is the runtime of Karatsuba's algorithm?

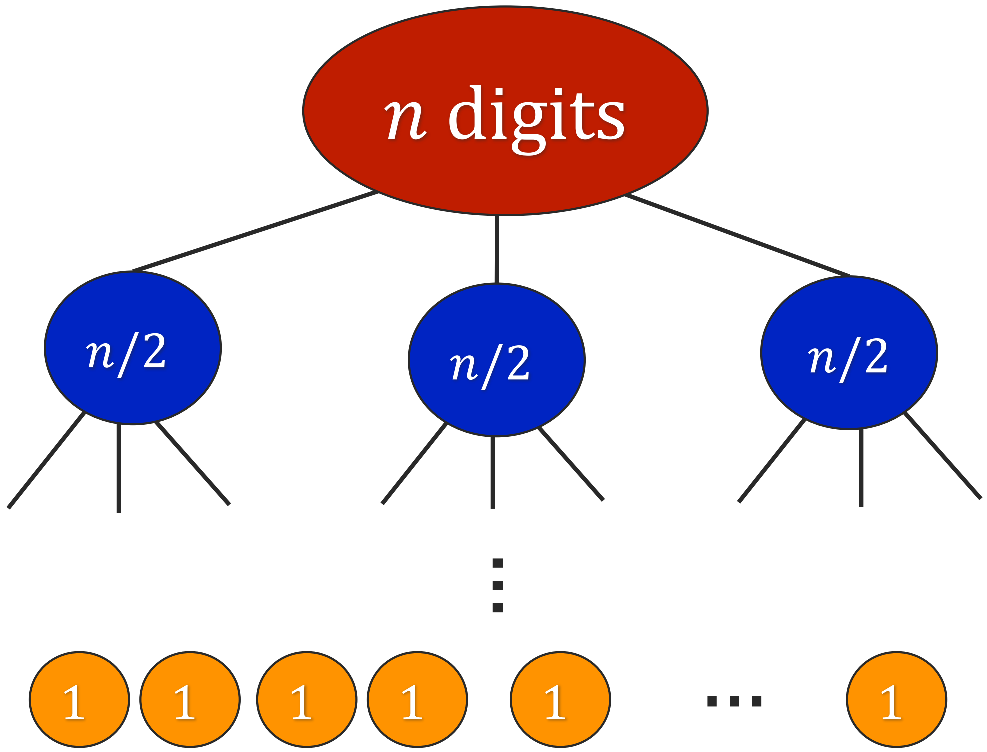
Less formally, how many 1-digit multiplications do we do in Karatsuba's algorithm?

Same approach as last lecture,

this time our branching factor is 3 instead of 4




| Layer | # of digits | # problems |
|---|---|---|
| 0 | $n$ | 1 |
| 1 | $n/2$ | 3 |
| ⋮ | ⋮ | ⋮ |
| $t$ | #digits per node | # nodes |
| ⋮ | ⋮ | ⋮ |
| Depth | 1 | # leaves |




| Layer | # of digits | # problems |
|---|---|---|
| 0 | $n$ | 1 |
| 1 | $n/2$ | 3 |
| ⋮ | ⋮ | ⋮ |
| $t$ | | |
| ⋮ | ⋮ | ⋮ |
| | 1 | |

# Other Algorithms

- **Karatsuba** (1960): $O(n^{1.6})$! Saw this!
- **Toom-3/Toom-Cook** (1963): $O(n^{1.465})$

(advanced)

Divide and conquer too! Instead of breaking into three n/2-sized problems, break into five n/3-sized problems.

**Hint:** Start with 9 subproblems and reduce it to 5 subproblems.

- **Schönhage–Strassen** (1971):
  - Runs in time $O(n \log(n) \log\log(n))$
- **Furer** (2007)
  - Runs in time $n \log(n) \cdot 2^{O(\log^*(n))}$
- **Harvey and van der Hoeven** (2019)
  - Runs in time $O(n \log(n))$

# What about binary representation?

We used base 10 so far

→Counted the # of 1-digit operations, assuming adding/multiplying single digits is easy (memorized our multiplication table!)

What if we use base 2?

→We would want to count # of 1-bit operations.

How do we alter Karatsuba's algorithm for binary numbers?

# N-bit integer multiplications

Easy to compute $10^k$ in base 10. In base 2, it is easy to compute $2^k$.

$$[b_1 b_2 \cdots b_n] = [b_1, b_2, \cdots, b_{n/2}] \times 2^{n/2} + [b_{n/2+1} b_{n/2+2} \cdots b_n]$$

$$\underbrace{b_1 \quad b_2 \quad b_3 \quad \cdots}_{b^L} \Big| \underbrace{\cdots b_{n-2} \quad b_{n-1} \quad b_n}_{b^R}$$

$$a \times b = (a^L \times b^L) 2^n + (a^L \times b^R + a^R \times b^L) 2^{n/2} + (a^R \times b^R)$$
$$= \cdots$$

**Practice:** Complete this equation the Karatsuba's way and rederived $O(n^{1.6})$ runtime for multiplying two $n$-bit numbers. Might see this on HW or Discussion!

# Details we skipped

Technically

- We only counted the number of 1-digit problems
- There are other things we do: adding, subtracting, …
- Shouldn't we account for all of that?

Absolutely!

- We should be more formal, and we will be next!
- In this case, additions/subtractions end up in lower order terms
- Don't affect $O(.)$.

# Asymptotic Notations More Formally

# Runtime of Algorithms Asymptotically

Suppose an algorithm with input size $n$ takes

$$T(n) = 5n^2 + 20n\log(n) + 7 \quad \text{ms}$$

$$T(n) \in O(n^2) \text{ also commonly written as } T(n) = O(n^2)$$

Why is it a good idea to just say this is $O(n^2)$?

- Constants like 5, 20, 7, depend on the platform and computer.
- Makes it easier to compare the performance of algorithms on large inputs
- Makes algorithm analysis easier
- Sometime clever tricks and representations improve the constants anyway.

# Definition of O(...)

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- We say "$T(n)$ is $O\left(g(n)\right)$" if and only if

  for large enough n,

  $T(n)$ is *at most* some constant multiple of $g(n)$.

# Definition of O(...)

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as  a runtime: positive and increasing in n.

- We say "$T(n)$ is $O\left(g(n)\right)$" if and only if

There exists $c$ and $n_0 > 0$

Such that for all $n \geq n_0, T(n) \leq c \cdot g(n)$

- Always give the tightest and simplest O( ) you can.

→ e.g., $5n^2 \in O(n^3)$ and $5n^2 \in O(2n^2 + n)$ too,

→ but give the best bound of $O(n^2)$.

# Example

Prove that for $T(n) = 2n^2 + 2$, we $T(n) \in O(n^2)$

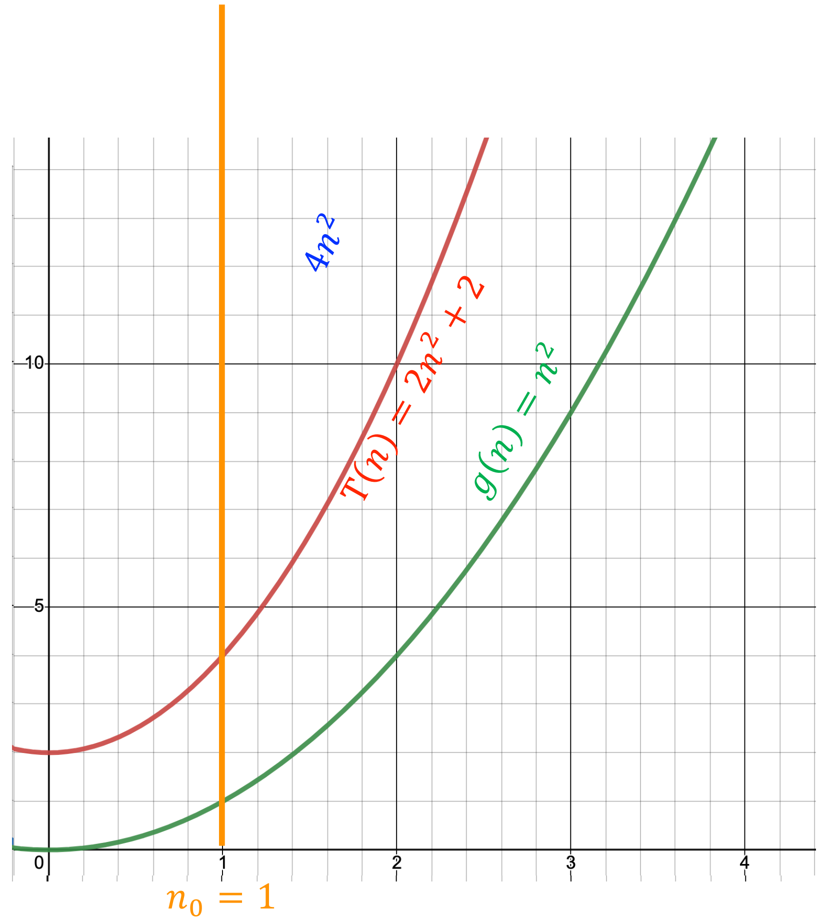
Even though $T(n)$ is larger than $n^2$ always, we can find $c = 4$ and $n_0 = 1$, such that all $n > n_0$

$$2n^2 + 2 \leq 4n^2$$

How do you prove the above inequality?

Whatever (correct!) math you like!

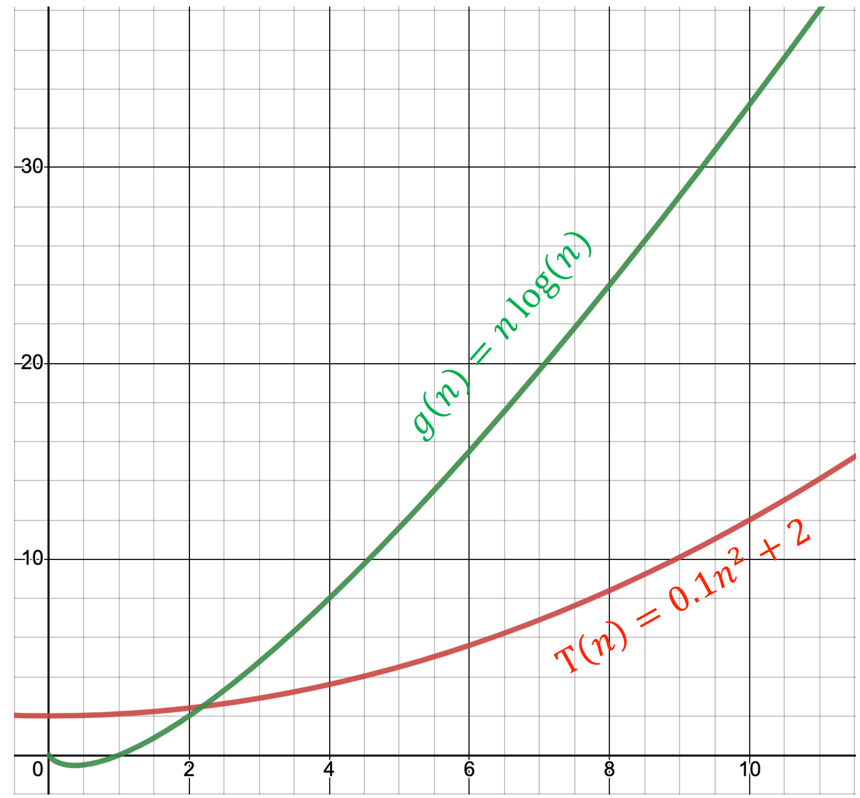
- E.g., equal at $n_0$ and RHS has larger derivative.



# BEWARE! of pictures!

The picture seems to imply that for $\mathrm{T}(n) = 0.1n^2 + 2$ we have that $\mathrm{T}(n) \in O(n\log(n))$!

What's wrong with this argument and relying on pictures?



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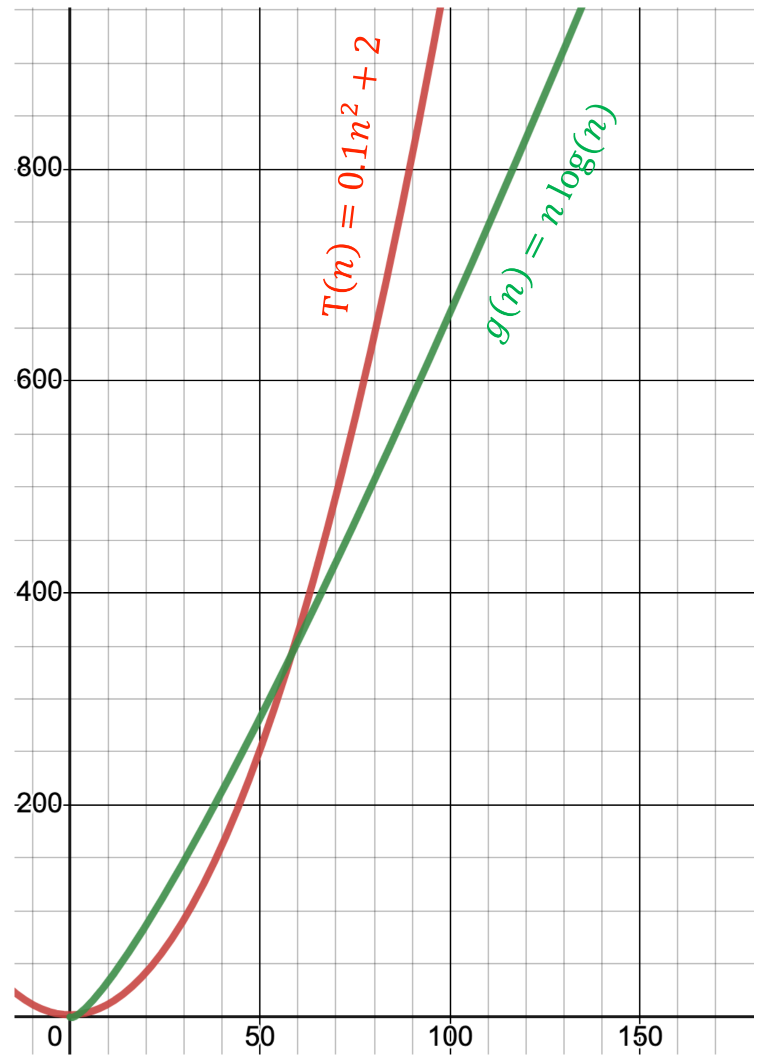



What's wrong with this argument and relying on pictures?

That is why you should come up with $c$ and $n_0$ and mathematically prove that for all $n \geq n_0, T(n) \leq c \cdot g(n)$.

# How to prove $0.1n^2 \notin O(n)$!

- Proof by contradiction:
- Suppose that $0.1\, n^2 \in O(n)$.
- Then there is some positive $c$ and $n_0$ so that:

$$\forall n \geq n_0, \qquad 0.1n^2 \leq c\, n$$

- Divide both sides by $n$:

$$\forall n \geq n_0, \qquad 0.1n \leq c$$

- That's not correct. Let $n = n_0 + 10\,\mathrm{c}$
  - Then $n \geq n_0$, but $0.1n > c$.
- Contradiction!

# Recap of Proof Techniques

To prove $T(n) \in O(g(n))$:

- You have to come up with $c$ and $n_0$ so that the definition is satisfied.

To prove $T(n) \notin O(g(n))$

- You have to rule out **all possible** $c$ and $n_0$.
- One approach is to use **proof by contradiction**:
  - Suppose there exists a $c$ and an $n_0$ so that the definition ***is*** satisfied.
  - Derive a contradiction,

  → e.g., by finding large enough $n$ (as a function of $c$ and $n_0$), for which the definition is not satisfied.

# $\Omega(\ldots)$ means lower bound

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.

- We say "$T(n) \in \Omega(g(n))$" if and only if

$$\text{There exists } c \text{ and } n_0 > 0$$

$$\text{Such that for all } n \geq n_0, c \cdot g(n) \leq T(n)$$

Switched these compared to O( )!!

# Example

Indeed, $0.1n^2 + 2 \in \Omega(n \log(n))$!

Prove this formally:

Find constants $c$ and $n_0 > 0$, such that for all $n \geq n_0$, $c\, n \log(n) \leq 0.1n^2 + 2$.

# $\Theta(\ldots)$ means both!

We say "$T(n)$ is $\Theta(g(n))$" iff both:

$$T(n) = O\big(g(n)\big)$$

and

$$T(n) = \Omega\big(g(n)\big)$$

# Example: Asymptotics of the geometric series

Take any constant $r$ and function $T(n) = 1 + r + r^2 + \cdots + r^n$

$$\text{Show that } T(n) = \begin{cases} \Theta(r^n) & \text{if } r > 1 \\ \Theta(1) & \text{if } r < 1 \\ \Theta(n) & \text{if } r = 1 \end{cases}$$

Prove formally at home!

Proof Idea: Recall sum of a geometric series that for $r \neq 1$:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Intuition:

- For $r > 1$, this is approximately $\frac{r^{n+1}}{r} = r^n$.
- For $r < 1$, $\frac{r^{n+1}-1}{r-1} \approx \frac{1}{1-r}$

Prove formally at home (also EX 0.2 of the book).

# Revisiting Karatsuba's Alg runtime, more formally

What is the runtime of Karatsuba's Alg?

At each layer, we have 3 problems
→ Each problem of size $\frac{n}{2}$.

**Karatsuba's Alg in 1 layer**

$Q1 = a \times c$ $Q2 = b \times d$ $Q3 = (a+b)(c+d)$

$x \times y = Q1 \times 10^{n} + (Q3 - Q1 - Q2) 10^{n/2} + Q2$

We have to do a bunch of other operations

- Finding a, b, c, d by shifting $n$-digit arrays.
- $n/2$-digit additions $a+b$, c+$d$
- $n$-digit additions $Q3 - Q1 - Q2$
- $2n$-digit additions $Q1 \times 10^{n} + (Q3 - Q1 - Q2) 10^{n/2} + Q2$
- ....

$O(n)$
More precisely $\leq 20n$

Recursive calls to the next layer

Work in this layer

Runtime : $T(n) = 3\,T\left(\frac{n}{2}\right) + 20n$

# Recurrence Relations

Recurrence relations give a formula for $T(n)$, i.e., the runtime on size $n$ problems in terms of $T(k)$ where $k < n$.

Recursive calls to the next layer

Work in this layer

$$\underbrace{}_{}T(n) = \overbrace{3\,T\left(\frac{n}{2}\right)}^{\text{Recursive calls to the next layer}} + \overbrace{20n}^{\text{Work in this layer}}$$ is a **recurrence relation.**

$T(1) = O(1)$ Base case (e.g., $T(1) = 5$ *or* 500)

Main question:

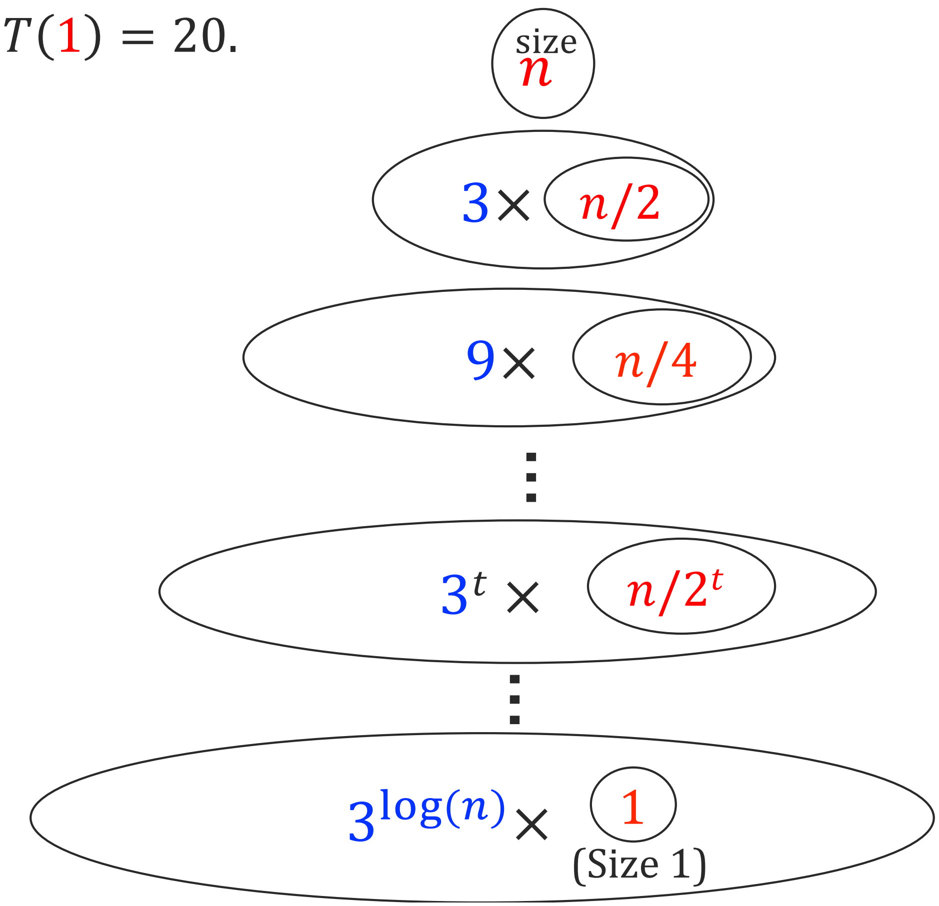
Given a recurrence relation for $T(n)$, find a closed-form expression for it.

For example, we hope that $T(n) = O(n^{1.6})$ for the above recurrence!

# Solve Karatuba's Alg Recurrence Relation

$$T(n) = 3T\left(\frac{n}{2}\right) + 20n, \quad T(1) = 20.$$

Abstraction of the tree




Total contribution in this layer

# Solve Karatuba's Alg Recurrence Relation

$$T(n) = 3T\left(\frac{n}{2}\right) + 20n, \quad T(1) = 20.$$

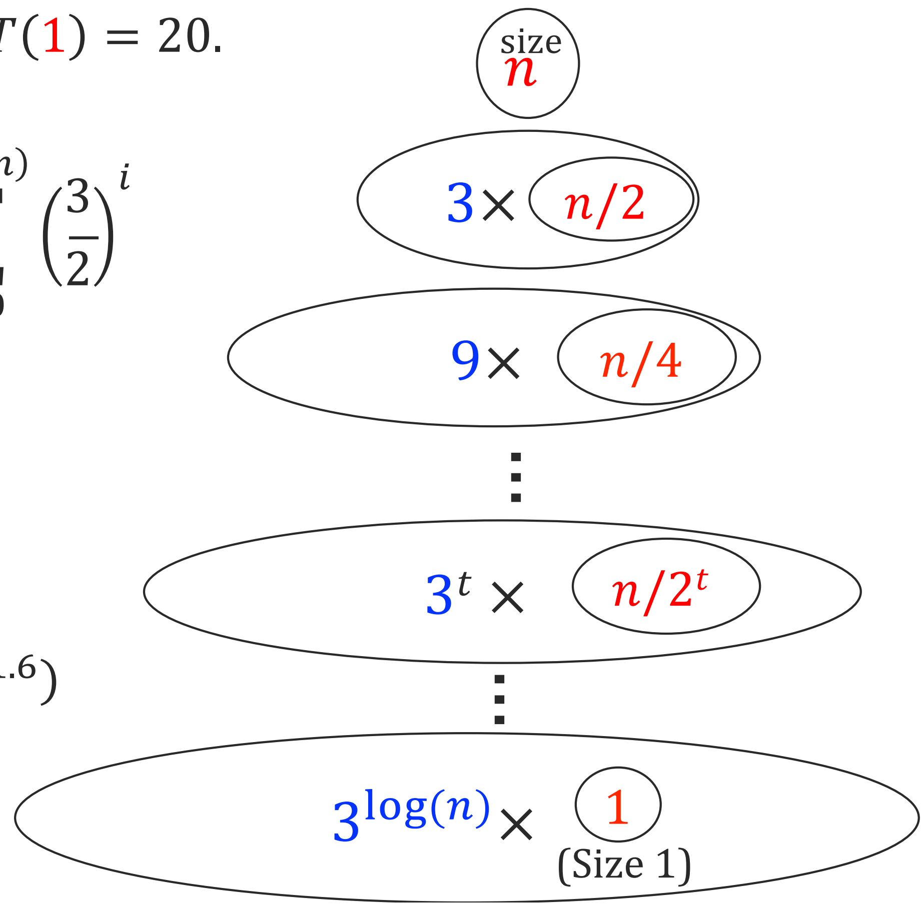
$$\sum_{i=0}^{\log(n)} 20n\left(\frac{3}{2}\right)^i = 20n\sum_{i=0}^{\log(n)}\left(\frac{3}{2}\right)^i$$

$$= O\left(n\left(\frac{3}{2}\right)^{\log(n)}\right)$$

$$= O\left(n \times n^{\log 3 - \log 2}\right)$$

$$= O\left(n^{\log(3)}\right) = O\left(n^{1.6}\right)$$

Abstraction of the tree




Total contribution in this layer

$20n$

$3 \times 20\left(\frac{n}{2}\right)$

$9 \times 20\left(\frac{n}{4}\right)$

$3^t \times 20\left(\frac{n}{2^t}\right)$

$3^{\log(n)} \times 20(1)$

# Solving Recurrence Relations Generally

The tree method, as we just did

- Keep track of the number and size of problems in each step
- Account for total amount of computation done in each layer.
- Sum over all the computation done in the layers.

# The Master Theorem

The tree method, as we just did

- Keep track of the number and size of problems in each step
- Account for total amount of computation done in each layer.
- Sum over all the computation done in the layers.

**The Master Theorem**

Suppose that $a \geq 1, b > 1,$ and $d \geq 0$ are constants (independent of n).
Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

# More on the Master Theorem

- Can it be used to solve any recurrence relation?

→ Nope! But it is a useful tool in many cases.

→ So, make sure you are also comfortable with the tree method.

- Don't we need a base case?

→Yes!

→Take $T(1) = O(1)$, the exact constant in this case doesn't affect the O(.).

- What if $n/b$ is not an integer?

→The Master Theorem is also correct with $T(n) = a \cdot T\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$.

→We will mostly **ignore floors and ceilings** in recurrence relations.

# Overview of the proof of Master Theorem

- See Section 2.2 of the book for a complete proof.

For the proof, suppose that $T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$.

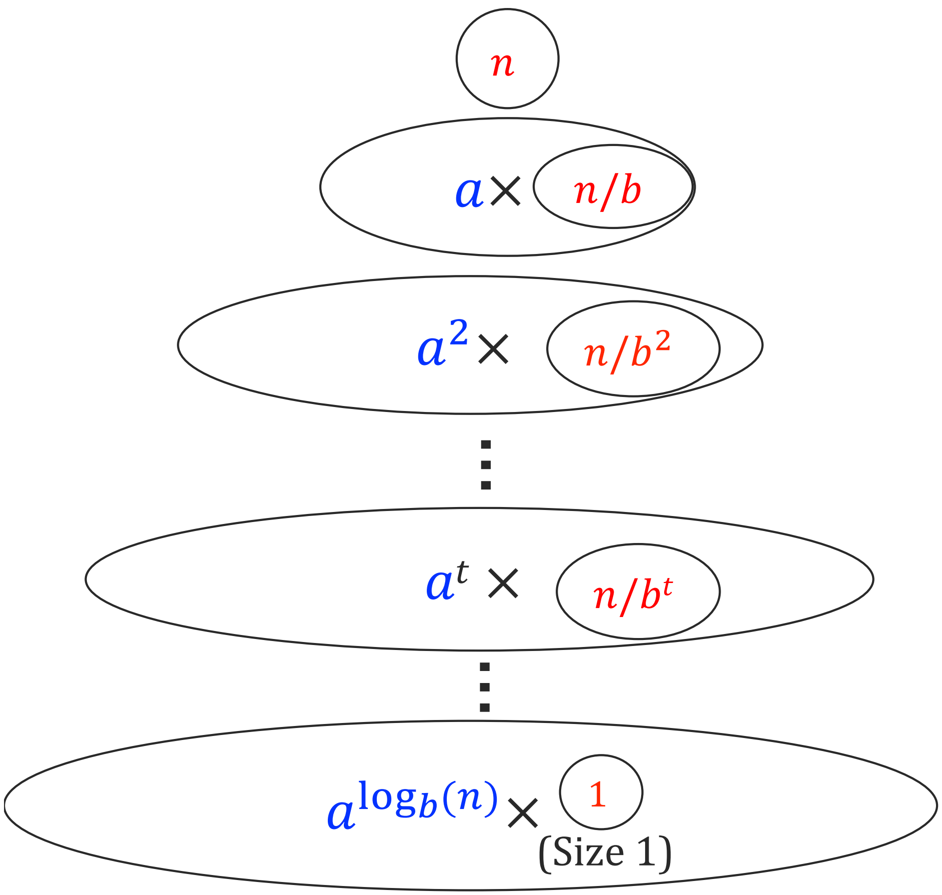
- For formal recursive arguments, we always substitute a constant.

→ Precise relationship between each layer's parameter and the amount of work.

→ Let's assume $T(1) = c$, too. For convenience!

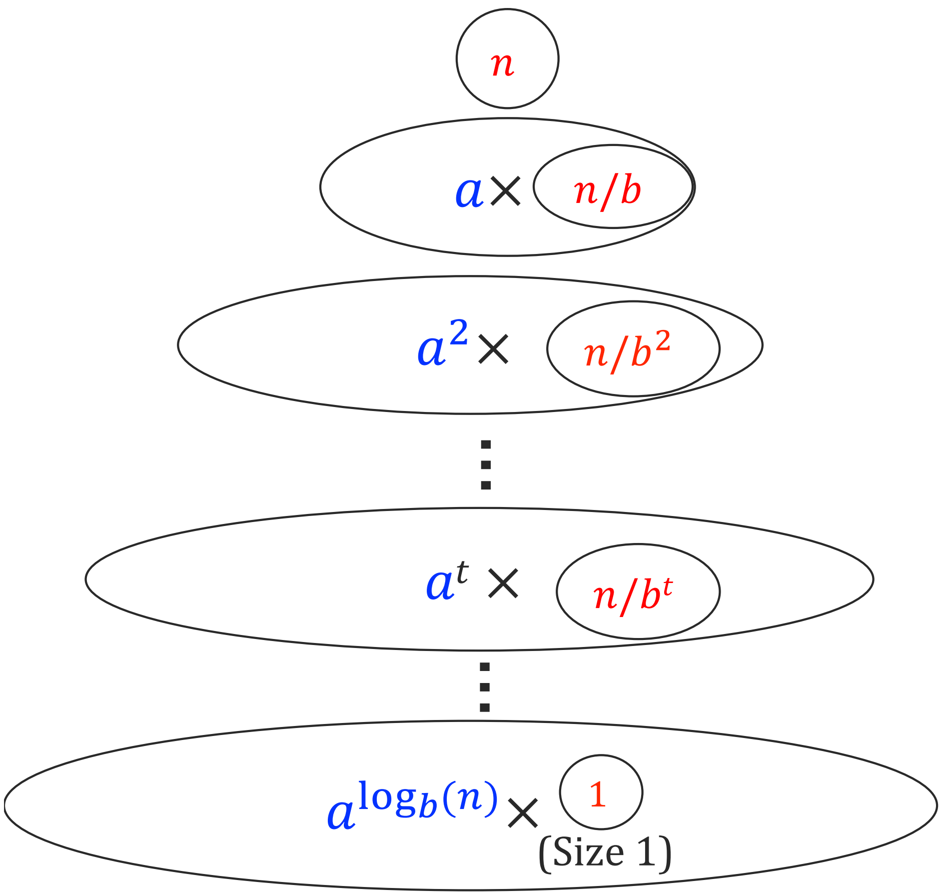
→ Just do the tree method!

$$T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$




| **Layer** | **Problem size** | **# problems** | **Work @ this layer** |
|---|---|---|---|
| 0 | $n$ | 1 | |
| 1 | $n/b$ | $a$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | |
| $t$ | $n/b^t$ | $a^t$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | |
| $\log_b(n)$ | 1 | $a^{\log_b(n)}$ | |

$$T(n) \le a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$




| Layer | Problem size | # problems | Work @ this layer |
|---|---|---|---|
| 0 | $n$ | 1 | $c \cdot n^d$ |
| 1 | $n/b$ | $a$ | $a \cdot c \cdot \left(\frac{n}{b}\right)^d$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $t$ | $n/b^t$ | $a^t$ | $a^t \cdot c \cdot \left(\frac{n}{b^t}\right)^d$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $\log_b(n)$ | 1 | $a^{\log_b(n)}$ | $a^{\log_b(n)} \cdot c$ |

$$T(n) \le a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Total computation on all layers:

$$cn^d \cdot \underbrace{\sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}_{\text{Looks so familiar ...}}$$

| Layer | Problem size | # problems | Work @ this layer |
|---|---|---|---|
| 0 | $n$ | 1 | $c \cdot n^d$ |
| 1 | $n/b$ | $a$ | $a \cdot c \cdot \left(\frac{n}{b}\right)^d$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $t$ | $n/b^t$ | $a^t$ | $a^t \cdot c \cdot \left(\frac{n}{b^t}\right)^d$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $\log_b(n)$ | 1 | $a^{\log_b(n)}$ | $a^{\log_b(n)} \cdot c$ |

# Proof of the Master Theorem

$$T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Total computation on all layers:

$$cn^d \cdot \underbrace{\sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}_{\text{Geometric series}}$$

Geometric series

$$1 + r + r^2 + \cdots + r^n = \begin{cases} \Theta(r^n) & \text{if } r > 1 \\ \Theta(1) & \text{if } r < 1 \\ \Theta(n) & \text{if } r = 1 \end{cases}$$

# The Master Theorem

$$T(n) \le a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Total computation on all layers:

$$n^d \cdot n^{\log_b\left(\frac{a}{b^d}\right)} = n^{d+\log_b(a)-\log_b(b^d)} = n^{d+\log_b(a)-d} = n^{\log_b(a)}$$

$$cn^d \cdot \underbrace{\sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}_{\text{Geometric series}} = \begin{cases} \Theta\left(n^d \left(\frac{a}{b^d}\right)^{\log_b(n)}\right) & \text{if } a > b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log(n)) & \text{if } r = 1 \end{cases}$$

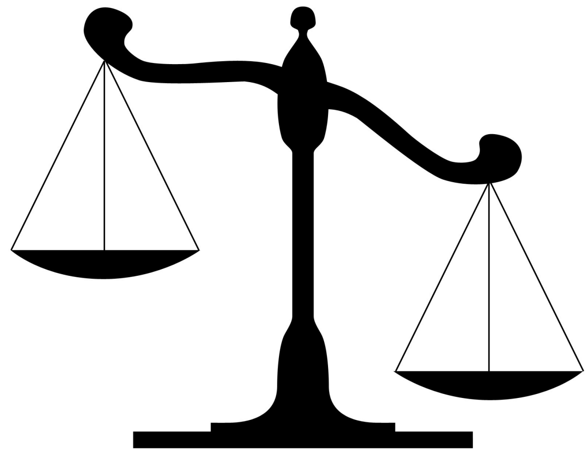
$$1 + r + r^2 + \cdots + r^n = \begin{cases} \Theta(r^n) & \text{if } r > 1 \\ \Theta(1) & \text{if } r < 1 \\ \Theta(n) & \text{if } r = 1 \end{cases}$$

# Master Theorem's Interpretation

$a$ vs. $b^d$

Wide tree
$a > b^d$

Branching causes the number of problems to explode!
**Most work is at the bottom of the tree!**



Tall and narrow
$a < b^d$

Problem size shrinks fast, so **most work is at the top of the tree!**

$a = b^d$
Branching perfectly balances total amount of work per layer.
**All layers contribute equally.**

# Wrap up

Karatsuba Integer Multiplication:

You can do better than grade school multiplication!
Example of divide-and-conquer in action
Runtime analysis, informal and formal.

Asymptotics, recurrence relations, and Master theorem

Tree method is intuitive and fun!
Master theorem is useful!

**Next time**

- More divide and conquer
- Matrix multiplications
- Median selection