Today: Quantum.
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Today: Quantum.

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Today: Quantum.
Qubit/electron.

\[ |0 \rangle \quad \text{ground state} \quad |1 \rangle \quad \text{excited state} \]

Superposition:

\[ \alpha_0 |0 \rangle + \alpha_1 |1 \rangle \]

Complex numbers \( \alpha_0 \) and \( \alpha_1 \).

\[ |\alpha_0|^2 + |\alpha_1|^2 = 1 \]

\( \alpha_0 \) and \( \alpha_1 \) are "amplitudes."
Qubit/electron.

ground state

\[ |0\rangle \]
Qubit/electron.

excited state

$|1\rangle$

Complex numbers $\alpha_0$ and $\alpha_1$.

$|\alpha_0|^2 + |\alpha_1|^2 = 1.$

$\alpha_0$, $\alpha_1$ are "amplitudes."
Qubit/electron.

- Ground state: $|0\rangle$
- Excited state: $|1\rangle$

Superposition: $\alpha |0\rangle + \alpha |1\rangle = 1$

$\alpha$ and $\alpha$ are "amplitudes."
Qubit/electron.

- **Ground state**: $|0\rangle$
- **Excited state**: $|1\rangle$
- **Superposition**: $\alpha_0 |0\rangle + \alpha_1 |1\rangle$

Complex numbers $\alpha_0$ and $\alpha_1$. 
Qubit/electron.

ground state \[ |0\rangle \]

excited state \[ |1\rangle \]

Superposition \[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \]

Complex numbers \( \alpha_0 \) and \( \alpha_1 \).
\[ |\alpha_0|^2 + |\alpha_1|^2 = 1. \]
Qubit/electron.

\[ |0\rangle \quad \text{ground state} \]
\[ |1\rangle \quad \text{excited state} \]

\[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \quad \text{Superposition} \]

Complex numbers \( \alpha_0 \) and \( \alpha_1 \).
\[ |\alpha_0|^2 + |\alpha_1|^2 = 1. \]
\( \alpha_0, \alpha_1 \) are “amplitudes.”
Measurement.

\[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \]
Measurement.

\[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \]

Remember \( |\alpha_0 |2 \rangle + |\alpha_1 |2 \rangle = 1 \).

Amplitudes \( \rightarrow \) probabilities on measurement!!
Measurement.

\[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \]

State \( |0\rangle \) with prob \( |\alpha_0|^2 \)
Measurement.

\[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \]

\[ \text{state } |0\rangle \text{ with prob } |\alpha_0|^2 \]

\[ \text{state } |1\rangle \text{ with prob } |\alpha_1|^2 \]
Measurement.

Remember $|\alpha_0|^2 + |\alpha_1|^2 = 1$. 

\[ |\alpha_0|^2 + |\alpha_1|^2 = 1. \]
Measurement.

\[ \alpha_0 |0\rangle + \alpha_1 |1\rangle \]

- State \(|0\rangle\) with prob \(|\alpha_0|^2\)
- State \(|1\rangle\) with prob \(|\alpha_1|^2\)

Remember \(|\alpha_0|^2 + |\alpha_1|^2 = 1\).

Amplitudes \(\rightarrow\) probabilities on measurement!!!
Two qubits...a dollar.

One bit:

Classic State: 0 or 1.

Quantum State: Internal:
\[ |\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle. \]

Measure: 0 or 1.

Two numbers internally, measurement yields one bit.

Two bits:

Classical State: 00, 01, 10, 11.

Quantum State: Internal:
\[ |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle. \]

Measure: 00, 01, 10, 11.

4 internal numbers, measurement yields two bits.

Ooh!

Something new, with two.

Partial Measure: look at one bit.

Result: 0 (with probability \[ |\alpha_{00}|^2 + |\alpha_{01}|^2 \].)

What is the state of the system if result is 0?

New Internal state:
\[ \alpha_{00} |00\rangle + \alpha_{01} |01\rangle \]

Scaling to make probabilities add to 1.
Two qubits...a dollar.

One bit:

**Classic State:** 0 or 1.
Two qubits...a dollar.

One bit:
**Classic State:** 0 or 1.
**Quantum State:**

Internal:
\[ |\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle. \]

Measure: 0 or 1.
Two numbers internally, measurement yields one bit.

Two bits:
**Classical State:** 00, 01, 10, 11.
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\[ |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{01} |10\rangle + \alpha_{11} |11\rangle. \]

Measure: 00, 01, 10, 11.
Four internal numbers, measurement yields two bits.

Ooh!
Something new, with two.
Partial Measure: look at one bit.
Result: 0 (with probability \[ |\alpha_{00}\rangle^2 + |\alpha_{01}\rangle^2. \])

What is the state of the system if result is 0?
New Internal state:
\[ \alpha_{00} |00\rangle + \alpha_{01} |01\rangle \]

\[ \sqrt{ |\alpha_{00}\rangle^2 + |\alpha_{01}\rangle^2} \]
Scaling to make probabilities add to 1.
Two qubits..a dollar.

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**Classic State:** 0 or 1.

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\[ |\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle. \]
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Ooh! Something new, with two.

Partial Measure: look at one bit. Result: 0 (with probability \[ |\alpha_0\rangle^2 + |\alpha_1\rangle^2. \])

What is the state of the system if result is 0?

New Internal state:
\[ \sqrt{\alpha_0^2 + \alpha_1^2} |00\rangle + \sqrt{\alpha_0^2 + \alpha_1^2} |01\rangle + \sqrt{\alpha_0^2 + \alpha_1^2} |10\rangle + \sqrt{\alpha_0^2 + \alpha_1^2} |11\rangle. \]
Two qubits..a dollar.

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**Classic State:** 0 or 1.

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Two numbers internally,
Two qubits... a dollar.

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Two numbers internally, measurement yields one bit.

Two bits:

**Classical State:** 00, 01, 10, 11.

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- Internal:

\[ |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{01} |10\rangle + \alpha_{11} |11\rangle. \]
- Measure: 00, 01, 10, 11.

4 internal numbers, measurement yields two bits.

Ooh!

Something new, with two.

Partial Measure: look at one bit.

Result: 0 (with probability \[ |\alpha_{00}\rangle^2 + |\alpha_{01}\rangle^2. \])

What is the state of the system if result is 0?

New Internal state:

\[ \alpha_{00} |00\rangle + \alpha_{01} |01\rangle \]

\[ \sqrt{ |\alpha_{00}\rangle^2 + |\alpha_{01}\rangle^2} = 1 \]

Scaling to make probabilities add to 1.
Two qubits...a dollar.

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**Classic State:** 0 or 1.
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Two numbers internally, measurement yields one bit.

Two bits:  
**Classical State:** 00, 01, 10, 11.
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  \[ |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle. \]
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Ooh! Something new, with two. 
Partial measure: look at one bit.
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What is the state of the system if result is 0?
New internal state: 
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Two numbers internally, measurement yields one bit.

Two bits:  
**Classical State:** 00, 01, 10, 11.
Two qubits...a dollar.

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Two numbers internally, measurement yields one bit.

Two bits:
**Classical State:** 00, 01, 10, 11.
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   Internal:
Two qubits..a dollar.

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Two numbers internally, measurement yields one bit.

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**Classical State:** 00, 01, 10, 11.

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\[ |\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1 \]
Two qubits...a dollar.

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**Classic State:** 0 or 1.

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- **Classical State**: 00, 01, 10, 11.
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  - Internal:
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    $|\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1$
  - Measure: 00, 01, 10, 11.

4 internal numbers,
Two qubits...a dollar.

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**Classic State:** 0 or 1.

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Two numbers internally, measurement yields one bit.

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$$|\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1$$

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Ooh!

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Measure : 00, 01, 10, 11.
4 internal numbers, measurement yields two bits.
Two qubits..a dollar.

One bit:

**Classic State:** 0 or 1.

**Quantum State:**
  - Internal: $\left| \alpha \right\rangle = \alpha_0 \left| 0 \right\rangle + \alpha_1 \left| 1 \right\rangle$.
  - Measure: 0 or 1.

Two numbers internally, measurement yields one bit.

  Ooh! Something new,

Two bits:

**Classical State:** 00, 01, 10, 11.

**Quantum State:**
  - Internal:
    $$\left| \alpha \right\rangle = \alpha_{00} \left| 00 \right\rangle + \alpha_{01} \left| 01 \right\rangle + \alpha_{01} \left| 10 \right\rangle + \alpha_{11} \left| 11 \right\rangle$$
    $$|\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1$$
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Internal:

\[ |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{01} |10\rangle + \alpha_{11} |11\rangle \]

\[ |\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1 \]

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Two qubits..a dollar.

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Two numbers internally, measurement yields one bit.

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**Classical State:** 00, 01, 10, 11.

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Internal:

\[ |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{01} |10\rangle + \alpha_{11} |11\rangle \]

\[ |\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1 \]

Measure: 00, 01, 10, 11.

4 internal numbers, measurement yields two bits.

Ooh! Something new, with two.

**Partial Measure:** look at one bit.
Two qubits..a dollar.

One bit:

**Classic State:** 0 or 1.

**Quantum State:**
- Internal: \( |\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \).
- Measure: 0 or 1.

Two numbers internally, measurement yields one bit.

Two bits:

**Classical State:** 00, 01, 10, 11.

**Quantum State:**
- Internal: \( |\alpha\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle \)
  \[ |\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1 \]
- Measure: 00, 01, 10, 11.

4 internal numbers, measurement yields two bits.

Ooh! Something new, with two.

**Partial Measure:** look at one bit.

Result: 0
Two qubits..a dollar.

One bit:
Classic State: 0 or 1.
Quantum State:
Internal: 
|α⟩ = α₀ |0⟩ + α₁ |1⟩.
Measure : 0 or 1.
Two numbers internally, measurement yields one bit.

Two bits:
Classical State: 00, 01, 10, 11.
Quantum State:
Internal: 
|α⟩ = α₀₀ |00⟩ + α₀₁ |01⟩ + α₀₁ |10⟩ + α₁₁ |11⟩
|α₀₀|² + ⋯ + |α₁₁|² = 1
Measure : 00, 01, 10, 11.
4 internal numbers, measurement yields two bits.

Ooh! Something new, with two.

Partial Measure: look at one bit.
Result: 0 (with probability |α₀₀|² + |α₀₁|².)
Two qubits..a dollar.

One bit:

**Classic State:** 0 or 1.

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\[ |\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle. \]

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Two numbers internally, measurement yields one bit.

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**Partial Measure:** look at one bit.

Result: 0 (with probability \[ |\alpha_{00}|^2 + |\alpha_{01}|^2. \])

What is the state of the system if result is 0?
Two qubits..a dollar.

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Internal:

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\[ |\alpha_{00}|^2 + \cdots + |\alpha_{11}|^2 = 1 \]

Measure: 00, 01, 10, 11.

4 internal numbers, measurement yields two bits.

Ooh! Something new, with two.

**Partial Measure:** look at one bit.

Result: 0 (with probability \( |\alpha_{00}|^2 + |\alpha_{01}|^2 \).)

What is the state of the system if result is 0?

**New Internal state:**

\[ \frac{\alpha_{00} |00\rangle + \alpha_{01} |01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \]
Two qubits..a dollar.

One bit:

**Classic State:** 0 or 1.

**Quantum State:**
- Internal: \( |\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \).
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Two numbers internally, measurement yields one bit.

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What is the state of the system if result is 0?

**New Internal state:** \( \frac{\alpha_{00} |00\rangle + \alpha_{01} |01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \)

Scaling to make probabilities add to 1.
Joint State: Entanglement

Qubit one internal state: $\alpha_0 |0\rangle + \alpha_1 |1\rangle$
Joint State: Entanglement

Qubit one internal state: $\alpha_0 |0\rangle + \alpha_1 |1\rangle$
Qubit two internal state: $\beta_0 |0\rangle + \beta_1 |1\rangle$
Joint State: Entanglement

Qubit one internal state: \( \alpha_0 |0\rangle + \alpha_1 |1\rangle \)
Qubit two internal state: \( \beta_0 |0\rangle + \beta_1 |1\rangle \)

Joint State: \( \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle \) 

Can all two bit states be decomposed?

Yes?
No?

No!

\( \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \)

Proof: Exercise 10.1

No solution to the system of four polynomial equations.

Product of \( \alpha_0 \beta_1 = 0 \) means one must be 0

"Bell State."

One key to the power of quantum.

Entanglement: measure the first bit as 0, the other bit is zero.

More complicated actually: Bell-CHSH inequalities.
Joint State: Entanglement

Qubit one internal state: \( \alpha_0 |0\rangle + \alpha_1 |1\rangle \)
Qubit two internal state: \( \beta_0 |0\rangle + \beta_1 |1\rangle \)

Joint State: \( \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle \),

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Can all two bit states be decomposed? Yes?
Joint State: Entanglement

Qubit one internal state: \( \alpha_0 |0\rangle + \alpha_1 |1\rangle \)
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Can all two bit states be decomposed? Yes? No?

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Proof: Exercise 10.1
Joint State: Entanglement

Qubit one internal state: \( \alpha_0 |0\rangle + \alpha_1 |1\rangle \)
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Joint State: \( \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle \),

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Proof: Exercise 10.1

No solution to the system of four polynomial equations.
Joint State: Entanglement

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No solution to the system of four polynomial equations.

Product of \( \alpha_0 \beta_1 = 0 \) means one must be 0 . . .
Joint State: Entanglement

Qubit one internal state: $\alpha_0 |0\rangle + \alpha_1 |1\rangle$
Qubit two internal state: $\beta_0 |0\rangle + \beta_1 |1\rangle$

Joint State: $\alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle$,

Can all two bit states be decomposed? Yes? No?

No! $\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$.

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More complicated actually: Bell-CHSH inequalities.
$n$-qubits.

Internal State: $\alpha_{0\cdots 0} |0\cdots 0\rangle + \alpha_{0\cdots 1} |0\cdots 1\rangle + \cdots + \alpha_{1\cdots 1} |1\cdots 1\rangle$. 
$\textit{n-qubits.}$

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Feynmann: how to simulate an $n$ particle system.
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Quantum Computer

Start with $n$ qubits,

Input $x$:  

$n$-bit string.
Quantum Computer

Input $x$: $n$-bit string.

Start with $n$ qubits, make superposition,

Exponential action $\rightarrow$ Factor in polynomial time!

Can't watch where the action happens. Measurement is random. This is ok, as long as answer is right with decent probability.

Why different than probability? After all, can generate lots of possibilities.

Partial measure changes remaining state. State $\equiv$ amplitudes.

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Start with $n$ qubits, make superposition, do some quantum op's, measure to get $n$ bits.
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Exponential action

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Input $x$: $n$-bit string.

Output $y$: $n$-bit string.
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Can add/subtract/scale amplitudes using Quantum gates.
   Not clear how to do it for probability.
Circuits.

Quantum Fourier Transform Circuit:

Input: \( \sum_{x \in \{0, 1\}^n} \alpha_x |x\rangle \).

Output: \( \sum_{x \in \{0, 1\}^n} \beta_x |x\rangle \).

Where \( \beta_x \) is Fourier Transform of \( \alpha_x \).

Note: \( n \)-qubit circuit, computing on \( 2^n \) amplitudes!

Randomized computations can't compute on probabilities.

Measurement: gives \( x \) with probability \( |\beta_x|^2 \).

No access to \( \beta_x \).

Just get index \( x \) with probability according to Fourier coefficient \( \beta_x \).

Quantum Fourier Sampling.

Actual output is one \( x \)!

Classical Functions: \( f(x) \)

Quantum Analog: copies input and computes \( f(x) \).

Input: \( \sum_{x \in \{0, 1\}^n} \alpha_x |x, 0\rangle \).

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Random computations are fine with this; same \( \alpha_x \).
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Classical/Quantum Circuit.

Classical

Quantum
Quantum Fourier Transform: more detail

$n$-qubits.

$\beta_0 \cdots \beta_0, \ldots, \beta_1 \cdots \beta_1$.  

Measure: get each $n$-bit string $y$ with probability $|\beta_y|^2$.  

Fourier Transform: multiplies by $M(\omega_2^n)$ with $O(n^2)$ gates.  

Size of circuit is polynomial in $n$.  

Gates act on all states in parallel.  

(like randomized computations.)  

Can compute (even subtract) with amplitudes!  

(which randomized computations can't do much.)  

FFT or multiply by $M(\omega_2^n)$ finds “period” of periodic input.
Quantum Fourier Transform: more detail

$n$-qubits.

$2^n$ amplitudes: $\alpha_{0\ldots0}, \ldots, \alpha_{1\ldots1}$. 
Quantum Fourier Transform: more detail

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QFT:

- Fourier transform of amplitudes: $\beta_{0...0}, \ldots, \beta_{1...1}$.
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$2^n$ amplitudes: $\alpha_0\ldots_0, \ldots, \alpha_1\ldots_1$.

QFT:

Fourier transform of amplitudes: $\beta_0\ldots_0, \ldots, \beta_1\ldots_1$.

Measure: get each $n$-bit string $y$ with probability $|\beta_y|^2$.

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Size of circuit is polynomial in $n$. 

Gates act on all states in parallel. (like randomized computations.)
Can compute (even subtract) with amplitudes! (which randomized computations can't do much.)

FFT or multiply by $M(\omega_{2^n})$ finds "period" of periodic input.
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Factoring and Roots of Unity

Factoring can be accomplished by finding non-negative square roots.
Factoring and Roots of Unity

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**Claim:** If $x$ is a non-trivial root of 1 modulo $N$ then $gcd(x + 1, N)$ is a non-trivial factor of $N$. 

Harder claim: If $N$ is an odd composite than for at least half of the $x$'s, either $gcd(x, N) \neq 1$ or the order $r$ of $x$ is even and $x^{r/2}$ is a nontrivial square root of 1 mod $N$.

Example: $4^2 \equiv 1 \pmod{15} \Rightarrow 4 - 1$ or $4 + 1$ are non-trivial factors of fifteen.

More generally: $x^2 \equiv 1 \pmod{15} \Rightarrow x^2 - 1 = (x + 1)(x - 1) = 0 \pmod{15}$. 
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**Example:** $15$

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More generally: $x^2 = 1 \pmod{15} \implies x^2 - 1 = (x + 1)(x - 1) = 0 \pmod{15}.$
Initialize with state:

\[
\sum_{m=0}^{M-1} a_m = 0 |a, 0\rangle
\]

Compute:

\[
\sum_{m=0}^{M-1} a_m = 0 |a, f(a)\rangle, f(a) = x^a
\]

Measure second register:

First register now has period \( r \)!

Claim:

Resulting \( \alpha \) has nonzero amplitudes with period \( r \).

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x^a = z \text{ for } a = j, j+r, j+2r, \ldots \text{ since } x^r = 1.
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Claim:

QFT of period \( k \) signal = \( \Rightarrow \) periodic signal of \( M/k \) with 0 shift!

Do several times:

Run and measure QFT output, Result is multiple of \( M/(r) \).

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Order of \( x \) is \( r \).

Check GCD \( (N, x^r/2 + 1) \).

Details: need period \( r \) to divide \( M \).

What is \( M \)?

2\( n \).

Need more sophisticated analysis...

but same ideas.
Initialize with state: \[ \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle \]
Roots and Unity and Fourier Transform

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\[x^a = z \text{ for } a = j, j + r, j + 2(r), \ldots\]
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Do several times:
Roots and Unity and Fourier Transform

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Do several times:
Run and measure QFT output,
Initialize with state: \( \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle \)

Compute: \( \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, f(a)\rangle, \quad f(a) = x^a \)

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Do several times:
- Run and measure QFT output,
  - Result is multiple of \( M/(r) \).
Initialize with state: \[ \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle \]

Compute: \[ \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, f(a)\rangle, \quad f(a) = x^a \]

Measure second register: first register now has period \( r \)!

Claim: Resulting \( \alpha \) has nonzero amplitudes with period \( r \).
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Do several times:
   - Run and measure QFT output,
     - Result is multiple of \( M/(r) \).

Compute GCD of results:
Roots and Unity and Fourier Transform

Initialize with state: \[ \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle \]

Compute: \[ \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, f(a)\rangle, \quad f(a) = x^a \]

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Do several times:
- Run and measure QFT output,
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Compute GCD of results: will likely be \( M/(r) \).
Initialize with state: $\frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle$

Compute: $\frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, f(a)\rangle$, $f(a) = x^a$

Measure second register: first register now has period $r$!

Claim: Resulting $\alpha$ has nonzero amplitudes with period $r$.
$x^a = z$ for $a = j, j + r, j + 2(r), \ldots$ since $x^r = 1$.

Claim: QFT of period $k$ signal $\implies$ periodic signal of $M/k$ with 0 shift!

Do several times:
  Run and measure QFT output,
    Result is multiple of $M/(r)$.

Compute GCD of results: will likely be $M/(r)$.
Order of $x$ is $r$.  

Roots and Unity and Fourier Transform

Initialize with state: $$\frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle$$

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Measure second register: first register now has period $$r$$!

Claim: Resulting $$\alpha$$ has nonzero amplitudes with period $$r$$.
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Do several times:
- Run and measure QFT output,
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Order of $$x$$ is $$r$$. Check GCD($$N, x^{r/2} + 1$$).
Initialize with state: \( \frac{1}{\sqrt{M}} \sum_{a=0}^{M-1} |a, 0\rangle \)

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Details: need period \( r \) to divide \( M \).
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  - Need more sophisticated analysis...but same ideas.
Mini-Conclusion.

Quantum Fourier Transform $\implies$ Factoring!
What’s a gate look like?

Hadamard Gate.

\[
\begin{align*}
|0\rangle & \xrightarrow{\text{H}} \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \\
|1\rangle & \xrightarrow{\text{H}} \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle
\end{align*}
\]
What’s a gate look like?

Hadamard Gate.

\[
\begin{align*}
|0\rangle & \rightarrow \sqrt{\frac{1}{2}}|0\rangle + \sqrt{\frac{1}{2}}|1\rangle \\
|1\rangle & \rightarrow \sqrt{\frac{1}{2}}|0\rangle - \sqrt{\frac{1}{2}}|1\rangle
\end{align*}
\]

Two bits.
What's a gate look like?

Hadamard Gate.

\[ |0\rangle \rightarrow \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \quad \quad |1\rangle \rightarrow \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \]

Two bits.

\[ H(\alpha_0 |0\rangle + \alpha_1 |1\rangle) \]
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Hadamard Gate.

\[ |0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \quad |1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \]

Two bits.

\[ H(\alpha_0 |0\rangle + \alpha_1 |1\rangle) = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |0\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |1\rangle. \]

Notice: added amplitudes and even subtracted amplitudes! Not so easy or even possible with probability.

Hadamard: Reflection over line at angle \( \frac{\pi}{8} \) on the \((x, y)\)-plane.

Controlled Not Gate.

\[ |00\rangle |00\rangle |10\rangle |11\rangle \]

Note: Operating on \( \alpha_0 |0\rangle + \alpha_1 |1\rangle \) + \( \alpha_0 |0\rangle + \alpha_1 |1\rangle \).

One gets \( \alpha_0 |0\rangle + \alpha_1 |0\rangle + \alpha_0 |1\rangle + \alpha_1 |1\rangle \).
What’s a gate look like?

Hadamard Gate.

\[
\begin{align*}
|0\rangle & \rightarrow \sqrt{2} \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \\
|1\rangle & \rightarrow \sqrt{2} \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right)
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**Hadamard Gate.**

\[
|0\rangle \quad \xrightarrow{\text{H}} \quad \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \\
|1\rangle \quad \xrightarrow{\text{H}} \quad \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle
\]

Two bits.

\[
H(\alpha_0 |0\rangle + \alpha_1 |1\rangle) = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |0\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |1\rangle.
\]

Notice: added amplitudes and even subtracted amplitudes!

Not so easy or even possible with probability.
What’s a gate look like?

Hadamard Gate.

\[
\begin{align*}
|0\rangle & \rightarrow H |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \\
|1\rangle & \rightarrow H |1\rangle - \frac{1}{\sqrt{2}} |0\rangle
\end{align*}
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Hadamard: Reflection over line at angle \(\pi/8\) on the \((x, y)\)-plane.
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**Hadamard Gate.**

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\begin{align*}
|0\rangle & \xrightarrow{H} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\
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Two bits.

\[H(\alpha_0|0\rangle + \alpha_1|1\rangle) = \frac{\alpha_0 + \alpha_1}{\sqrt{2}}|0\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}}|1\rangle.\]

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\[x = \alpha_0, \quad y = \alpha_1.\]
What’s a gate look like?

**Hadamard Gate.**

\[ |0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \quad |1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \]

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\[ H(\alpha_0 |0\rangle + \alpha_1 |1\rangle) = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |0\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |1\rangle. \]

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Hadamard: Reflection over line at angle \(\pi/8\) on the \((x, y)\)-plane.

\[ x = \alpha_0, \quad y = \alpha_1. \]

**Controlled Not Gate.**

\[ |00\rangle \quad |01\rangle \quad |10\rangle \quad |11\rangle \]

Note:

Operating on \(\alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle\).
What’s a gate look like?

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\begin{align*}
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H(\alpha_0 |0\rangle + \alpha_1 |1\rangle) = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |0\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |1\rangle.
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**Hadamard:** Reflection over line at angle \(\pi/8\) on the \((x, y)\)-plane.

\(x = \alpha_0, y = \alpha_1\).

**Controlled Not Gate.**

\[
\begin{align*}
|00\rangle & \xrightarrow{\oplus} |00\rangle \\
|10\rangle & \xrightarrow{\oplus} |11\rangle
\end{align*}
\]

Note:

Operating on \(\alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle\).

One gets \(\alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{11} |10\rangle + \alpha_{10} |11\rangle\).
Quantum Fourier Transform.

Fourier Transform:

- Split into odd and even inputs.
- Recurse: 2 subcircuits.

Combine Input $x_0$ and $x_1$ from subcircuits.

Recursively compute $A_e$ and $A_o$ on $n^2$ roots of unity: $\omega_2, \omega_4, \omega_6, \ldots, \omega_n$.

For each $i \leq n/2$,

$$A(\omega_i) = A_e(\omega_2i) + \omega_i A_o(\omega_2i)$$

$$A(\omega_i + n/2) = A_e(\omega_2i) - \omega_i A_o(\omega_2i)$$

Runtime Recurrence:

$$T(n) = 2T(n/2) + O(n) = O(n \log n)!$$
Quantum Fourier Transform.

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Split into odd and even inputs.
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$A(\omega^i + n/2) = A e^{\omega^2 i} - \omega^i A o(\omega^2 i)$

Split: ignore low order bit.

The amplitudes of both will be processed in parallel.

Recurse: build one QFT circuit on $n-1$ bits.

The circuit will work on amplitudes of strings for both $x_0$ and $x_1$.

Combine: Add Hadamard Gate on $n$th bit.

Combines amplitudes of $x_0$ and $x_1$ in fancy way.

E.g. $\alpha_0 x^\pm \alpha_1 x$ plus scaling.

Note: need to do more than combine, need to multiply some by $\omega^j$.

(Phase.)

See Book for details $O(n)$ gates for phase multiplication.

Use conditional phase gates in construction.

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And that was quantum..
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And the semester.
And that was quantum..

And the semester.

It is a total privilege teaching you!!!
And that was quantum..

And the semester.

It is a total privilege teaching you!!!
    From me, Professor Raghavendra and the whole staff.
And the semester.

It is a total privilege teaching you!!!
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Good luck (skill) on Final...
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