Today.

...Complex numbers, polynomials today. FFT.

Multiplying polynomials.

(1+ 2x + 3x^2)(4+ 3x + 2x^2)
Coefficient of x^4 in result?
(A) 6
(B) 5

(A) 6 of course!
Coeefficient of x^2 in result?
Uh oh...

Hmmm...

Multiplying polynomials.

(1 + 2x + 3x^2)(4 + 3x + 2x^2)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Formula</th>
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</thead>
<tbody>
<tr>
<td>x^0</td>
<td>((1)(4))</td>
</tr>
<tr>
<td>x^1</td>
<td>((1)(3) + (2)(4))</td>
</tr>
<tr>
<td>x^2</td>
<td>((1)(2) + (2)(3) + (3)(4))</td>
</tr>
<tr>
<td>x^3</td>
<td>(2)(2) + (3)(3)</td>
</tr>
<tr>
<td>x^4</td>
<td>(3)(2)</td>
</tr>
</tbody>
</table>

4 + 11x + 20x^2 + 13x^3 + 6x^4

Given: a_0 + a_1x + ··· a_dx^d In example: a_0 = 1, a_1 = 2, a_2 = 3
b_0 + b_1x + ··· b_dx^d In example: b_0 = 4, b_1 = 3, b_2 = 2
Product: c_0 + c_1x + ··· c_dx^d

\[ c_k = \sum_{0 \leq i \leq k} a_i b_{k-i} \]

E.g.: c_2 = a_0b_2 + a_1b_1 + a_2b_0.

Another representation.

Represent a line?
Slope and intercept! a_0, a_1
How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3
How many points determine a degree d polynomial?

d + 1

How to find points on function?
plug in x-values...and evaluate.

How to find “line” from points?
Solve two variable system of equations!

How to find polynomial from points?
Solve d + 1 variable system of equations!
Evaluation of polynomials: Recursive.  

\[ A(x) = A_d(x^2) + x(A_{d-1}(x^2)) \]

where

Even coefficient polynomial.

\[ A_d(x) = a_0 + a_1x + a_2x^2 + \cdots \]

Odd coefficient polynomial.

\[ A_{d-1}(x) = a_1 + a_3x + a_5x^2 + \cdots \]

Example:

\[ A(x) = 4 + 12x + 20x^2 + 13x^3 + 6x^4 + 7x^5 = (4 + 20x + 6x^2) + (12x + 13x^3 + 7x^5) = (4 + 20x + 6x^2) + x(12 + 13x^3 + 7x^5) \]

\[ A_d(x) = 4 + 20x + 6x^2 \]

\[ A_{d-1}(x) = 12 + 13x + 7x^2 \]

Evaluate recursively:

For a point \( x \):

Compute \( A_d(x^2) \) and \( A_{d-1}(x^2) \).

\[ T(n) = 2T(n/2) + 1 = O(n) \]

\( O(n) \) for 1 point!

\( n \) points \( \Rightarrow O(n^2) \) time to evaluate on \( n \) points.

No better than polynomial multiplication! Bummer.
Explore recursion.

Recursive Condition:

- **n** points: \( \frac{n}{2} \) pairs of distinct numbers with common squares.
- E.g., \( \pm x_0 \) both have \( x_0^2 \) as square,
  \( \pm x_1 \) both have \( x_1^2 \) as square.

Next step:

- \( \frac{n}{2} \) points: squares should only be \( \frac{n}{4} \) distinct numbers
- But all our \( \frac{n}{2} \) points are squares ...and positive!

Need the \( \frac{n}{2} \) points to come in Positive/Negatives pairs!

How can squares be negative?

Complex numbers!

Pairs with common squares.

Want \( n \) numbers:
- \( x_0, \ldots, x_n \) where
  \[ |\{x_0^2, \ldots, x_n^2\}| = \frac{n}{2}, \]
- and
  \[ |\{x_0^2, \ldots, x_n^2\}| = \frac{n}{4}. \]

...and...

\[ \{x_0^2, \ldots, x_n^2\} = \{1, -1, i, -i\}. \]

Each recursive level evaluates:
- polynomials of half the degree on half as many points.
- \( n \) represents both degree and number of points.

In reverse: start with a number 1

Take square roots: 1, \(-1\), 1

Take square roots: 1, \(-1\), \(i\)

Uhh oh.

Actually: \( \pm 1, \pm i, \pm \frac{1}{\sqrt{2}}(1+i), \pm \frac{1}{\sqrt{2}}(-1+i) \),

Complex numbers!

Uhh oh? Can we get a pattern?

The \( n \)th complex roots of unity.

\[ (e^{2i\pi/n})^2 = (e^{2i\pi/n})^2 = e^{2i\pi}. \]

Solutions to \( z^n = 1 \)

\[ (1, 2\pi/n)^n = (1, 2\pi/n)^n = (1, 0)^n. \]

Pair w/common square!

Squares: \( \frac{n}{2} \)th roots.

Multiplying Complex Numbers

- \( r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \)
- \( r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1} \)
- \( r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2} \)
- \( r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) = (r_1r_2)e^{i(\theta_1 + \theta_2)} \)

Recursion on more than one point.

\[ A(x) = A_0(x^2) + x(A_0(x^2)) \]

Reuse computations.

- \( n \) points:
  \[ \pm x_0, \pm x_1, \ldots, \pm x_{n/2}. \]
  \[ \text{Also } n = \text{d + 1: number of coefficients.} \]

Two points: \( \pm x_0 \) and \( -x_0 \)

- One square: \( (\pm x_0)^2 = (-x_0)^2 = x_0^2. \)

- \( A(x_0) = A_0(x_0^2) + x_0A_0(x_0^2) \)

- \( A(-x_0) = A_0((-x_0)^2) + (-x_0)A_0((-x_0)^2) \)

- \( A(-x_0) = A_0(x_0^2) - x_0A_0(x_0^2) \)

From \( A_0(x_0^2) \) and \( A_0(x_0^2) \) compute both \( A(-x_0) \) and \( A(x_0) \)?

From \( A_0(x_0^2) \) and \( A_0(x_0^2) \) compute both \( A(-x_0) \) and \( A(x_0) \)?

Evaluate \( n \) coefficient polynomial on \( n \) points by

Evaluating \( \frac{n}{2} \) coefficient polynomials on \( \frac{n}{2} \) points.

\[ T(n,n) = 2T(\frac{n}{2}, \frac{n}{2}) + O(n) = O(n\log n) !!!! \]

From \( O(n^2) \) to \( O(n\log n) !!! \)

Can we get a pattern?

Pairs with common squares.

- \( A(x) = A_0(x^2) + x(A_0(x^2)) \)

- \( r = \sqrt{a^2 + b^2} \)

- \( \theta = \tan^{-1}(\frac{b}{a}) \)

- Polar coordinate: \( r(\cos\theta + i\sin\theta) = re^{i\theta} \) or \( (r, \theta) \)

Explore recursion.

Recursive Condition:

- \( n \) points: \( \frac{n}{2} \) pairs of distinct numbers with common squares.
- E.g., \( \pm x_0 \) both have \( x_0^2 \) as square,
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...and...

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Complex numbers!
Quiz

Which are the same as 1?
(A) (1)2
(B) (−1)2
(C) −1
(D) e2πi
(E) (eπi )2

Which are the same as −1?
(A) (1)2
(B) (−1)2
(C) −1
(D) e2πi
(E) (eπi )2

Note: e2πi = −1.

Which are 4th roots of unity? (Hint: take the 4th power.)
(A) eπi/2
(B) eπi
(C) eπi/3
(D) e3πi/2

The FFT!

Defn: ω = (1, 2π/n) = e2πi/n, nth root of unity.

Pairs: ωi and ωi+n/2. Common square!

Fast Fourier Transform:
Evaluate A(x) = a0 + a1x + a2x2 + ... + an−1xn−1 on points e2πi, e2πi2, ..., e2πi, 2.

Procedure:
Recursively compute A0 and A1 on 2 points of unity:
e2πi/2, e4πi/2, ..., enπi/2.

For each j ≤ n/2:
A(ωj) = A0(ωj2) + ωjA1(ωj2)
A(ωj+n) = A0(ωj+n) − ωjA1(ωj+n)

Runtime Recurrence:
A0 and A1 are degree n/2, n/2 points in recursion.
T(n) = 2T(n/2) + O(n) = O(nlog n)!

Quiz 2: review

What is ω2n/2?
What is (ωn)2n/2?
What is (eπi/2)n2?

Consider n points: Sn = {a0, a1, ..., an−1}.

How many points in the set: {(a0)n, (a1)n, ..., (an−1)n}?
n/2 points!!!

FFT: Evaluate degree n polynomial on n points
by evaluating two degree n/2 polynomials on n/2 points!

Summary.

Polynomial Multiplication: O(n2).
In Point form: O(n).

Polynomial Evaluation: O(n2).
Polynomial: A(x) = A0(x2) + xA1(x2)
Evaluate on n points recursively.
T(n, n) = 2T(n/2, n) + O(n) = O(n2).
The number of leaves is n.
and the work on each leaf is O(n).

Consider n points: Sn = {a0, a1, ..., an−1}.
Set of squares: Sn/2 = {a02, a12, ..., (an−1)2}.
Set of squares: Sn/2 = {a0n, a1n, ..., (an−1)n}.
Only n/2 values here.

Evaluate A(x) = A0(x2) + xA1(x2).
Only need to evaluate A0 and A1 on n/2 points.
T(n, n) = 2T(n/2, n) + O(n).
Or T(n) = 2(n/2) + O(n) = O(nlog n)