CS 170
Efficient Algorithms and Intractable Problems

Lecture 4:
Divide and Conquer III

Nika Haghtalab and John Wright

EECS, UC Berkeley

Announcements

How are discussions going?

Homework party

- Tomorrow (Friday), 10-2 @ Cory courtyard

Nika's office hour for next week:

- No after-class OH on Tuesday --- Lecture will be by Prof. John Wright.
- Instead Monday (Feb 3) 1:15-2pm @Cory courtyard

Recap of last lecture

Matrix Multiplication:

Strassen's algorithm

Similar to Karatsuba, we reduce the number of subproblems from 8 to 7.

(Median) Selection

We saw that a good pivot selection gives us O(n)

This lecture

- Continue with Median selection

→Formalize the discussion of pivots and prove O(n) runtime

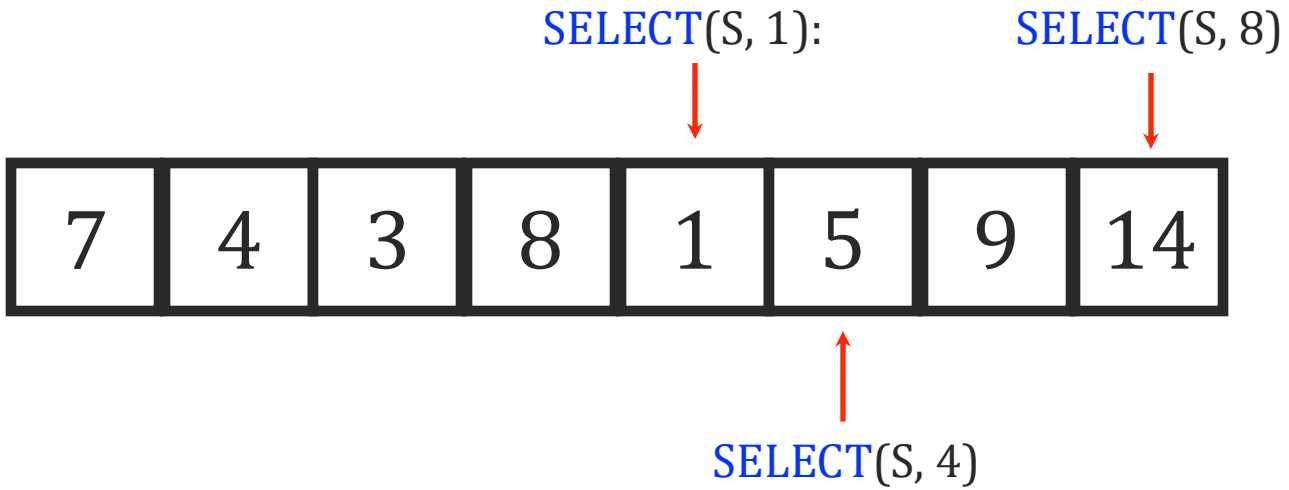
- One last example of Divide and Conquer: the Closest Pair problem

(Median) Selection

Recap: The k-select Problem

Input: Given an array S of n numbers and $k \in \{1, 2, \ldots, n\}$,

Output: Find the kth smallest element of S.



Some special cases:

SELECT(S, 1): Minimum element of the array

SELECT(S, n): Maximum element of the array

SELECT(S, $\left\lceil \frac{n}{2} \right\rceil$): Median element of the array

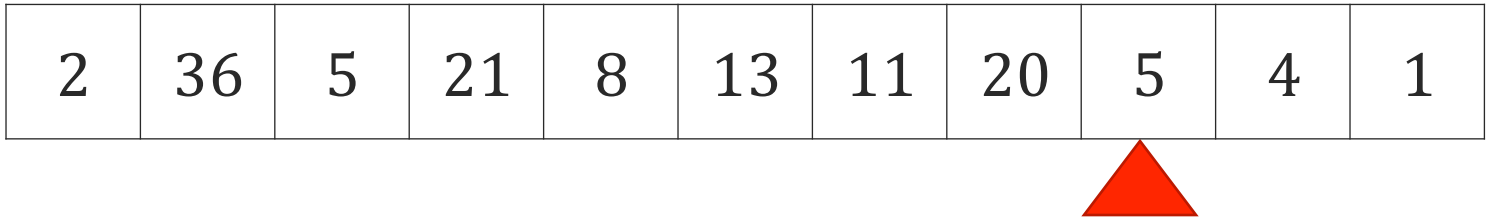
Recap: Big Question

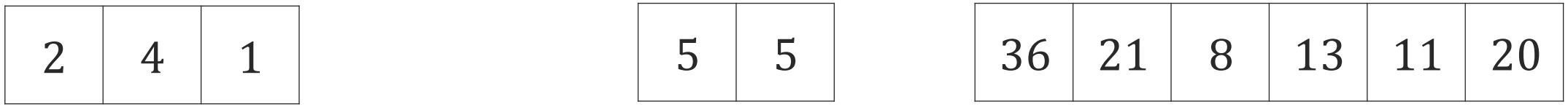
Can we perform Median selection
(or any other k-select generally)
in $O(n)$?

Recap: Divide and Conquer for SELECT(S, k)

We want to divide the problem to subproblems. How?

- Given a **"pivot" v.** Split the array into three pieces





S_L: Elements less than the pivot

S_v: Elements equal to the pivot

S_R: Elements larger than the pivot

SELECT(S, k):

- If $k \leq len(S_L)$: Return SELECT(S_L, k)
- If $len(S_L) < k \leq len(S_L) + len(S_v)$: Return v.
- If $len(S_L) + len(S_v) < k$: Return SELECT$(S_R, k - len(S_L) - len(S_v))$

Recap: The Recurrence Relation

SELECT(S, k):

- If $k \leq len(S_L)$: Return SELECT(S_L, k)
- If $len(S_L) < k \leq len(S_L) + len(S_v)$: Return v.
- If $len(S_L) + len(S_v) < k$: Return SELECT$(S_R, k - len(S_L) - len(S_v))$

$$T(n) = \begin{cases} T\big(len(S_L)\big) + O(n) & \text{if } k \leq len(S_L) \\ T\big(len(S_R)\big) + O(n) & \text{if } len(S_L) + len(S_v) < k \\ O(n) & \text{if } len(S_L) < k \leq len(S_L) + len(S_v) \end{cases}$$

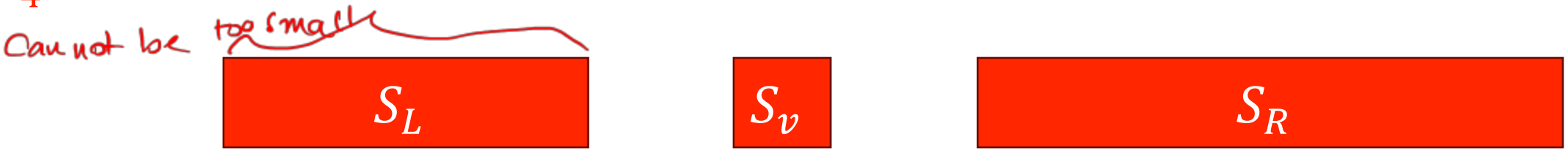
$$T(n) \leq T(\max\{len(S_L), len(S_R)\}) + O(n)$$

The lengths of S_L and S_R depend on the choice of the pivot.

Let’s formalize what is a ”good” pivot

Thought Exercise: "Good" Enough Pivot

Let's pretend that the pivot we picked is always between the $\frac{n}{4}th$ smallest and $\frac{3n}{4}$ th smallest element! What is the runtime of SELECT(S, k)?



Discuss

Assuming that the pivot is between the $\frac{n}{4}th$ smallest and $\frac{3n}{4}$ th smallest element, what's the maximum $\text{len}(S_L)$ and $\text{len}(S_R)$? $\leq \frac{3n}{4}$

$\text{len}(S_L) \geq n/4 \Rightarrow \text{len}(S_R) \leq \frac{3n}{4}$. Similarly for S_R.

Write down the recurrence relationship in this case:

$T(n) \leq T\left(\frac{3n}{4}\right) + O(n)$

Thought Exercise: "Good" Enough Pivot

Let's pretend that the pivot we picked is always between the $\frac{n}{4}th$ smallest and $\frac{3n}{4}$ th smallest element! What is the runtime of SELECT(S, k)?

So the runtime can be expressed by $T(n) \leq T\left(\frac{3n}{4}\right) + O(n)$

a=1 d=1 4/3

What's the runtime?

- $a = 1, b = 4/3, d = 1, a < b^d$
- $O(n)$ runtime.

The Master Theorem

Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

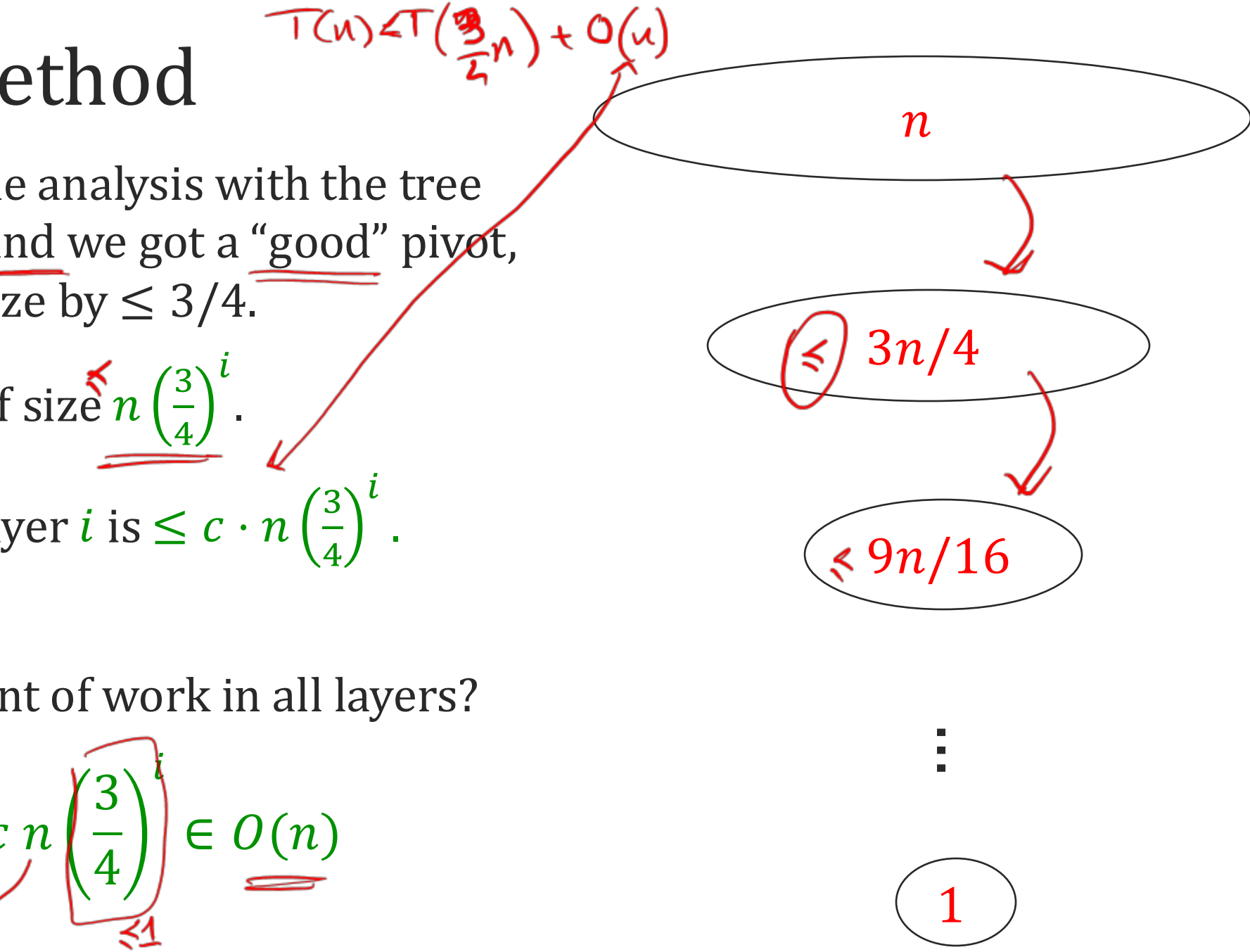
$$T(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log(n)) & \text{if } a = b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

In the Tree Method

Let's repeat the runtime analysis with the tree method: If in every round we got a "good" pivot, then we multiply the size by $\leq 3/4$.

Single node at layer i of size $n\left(\frac{3}{4}\right)^i$.

Total contribution at layer i is $\leq c \cdot n\left(\frac{3}{4}\right)^i$.



What is the total amount of work in all layers?

$$T(n) \leq \sum_{i=0}^{\log_{4/3}(n)} c\, n\left(\frac{3}{4}\right)^i \in O(n)$$

Another Thought Exercise on Good Pivots

Let's pretend that the pivot we picked is always between the $\frac{n}{10}$ th smallest and $\frac{9n}{10}$ th smallest element! What is the runtime of SELECT(S, k)?

Then, $len(S_R) \leq \frac{9n}{10}$ and $len(S_L) \leq \frac{9n}{10}$. So, the recurrence is $T(n) \leq T\left(\frac{9n}{10}\right) + O(n)$

a=1 d=1 b=10/9

In this case as well,

- $a = 1, b = 10/9, d = 1, a < b^d$
- $O(n)$ runtime!

The Master Theorem

Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

$$T(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log(n)) & \text{if } a = b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Summarizing Our Thoughts

Any pivot that's kind of in the middle is a good enough pivot

→ Call Pivots between the $\frac{n}{4}th$ smallest and $\frac{3n}{4}$ th smallest elements "good" pivots

→ There are more than half of the elements in the array are "good" pivots

→If we pick a random element, we have 50% chance of getting a "good" pivot

We don't need a "good" pivot at every round.

→We just need to get "good" pivots often enough

→ Every time we have a "good" pivot, the problem size shrinks to $\frac{3}{4}$ of the last one.

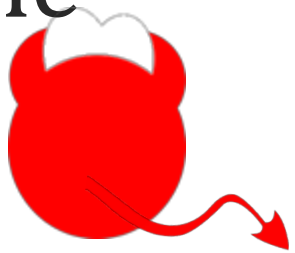
If we pick a random element as pivot, the expected runtime is fast!

There is randomized algorithm that

solves SELECT(S, k) in expected runtime of $O(n)$!

This algorithm is called QuickSelect and selects a uniformly random pivot on every turn.

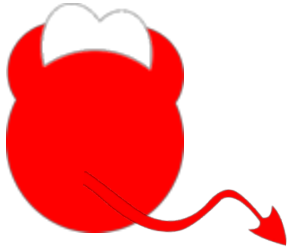
Randomized Algorithms and Expected Runtime



We typically think about runtime of an Alg on the **worst possible** problem instance.

Randomized Algorithms:

1. Write down the algorithm description.
2. Adversary sees the description and picks a bad instance.



3. Run the algorithm and throw the dice.

The adversary (choice of bad problem instance) doesn't depend on the randomness.

The running time is a **random variable.**

- It makes sense to talk about **expected running time**.

Expected Running Time and Divide and Conquer

We are interested in **expected runtime.**

$$\mathbb{E}[T(n)]$$

averages over runtimes $T(i)$ based on the probability of getting a subproblem of size i.

$\mathbb{E}[T(n)]$ is small when large size i has very low probability of happening

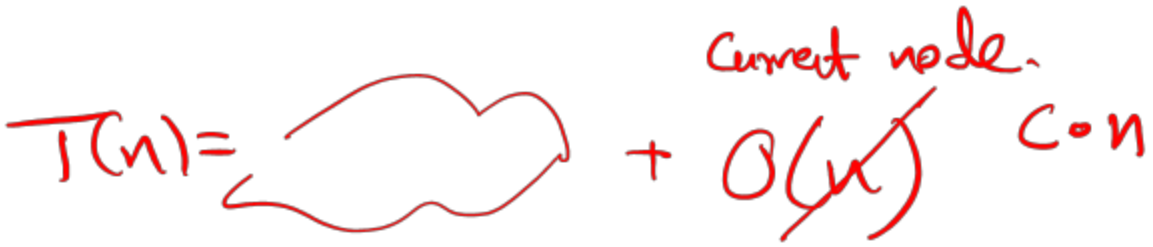
Trees Revisited

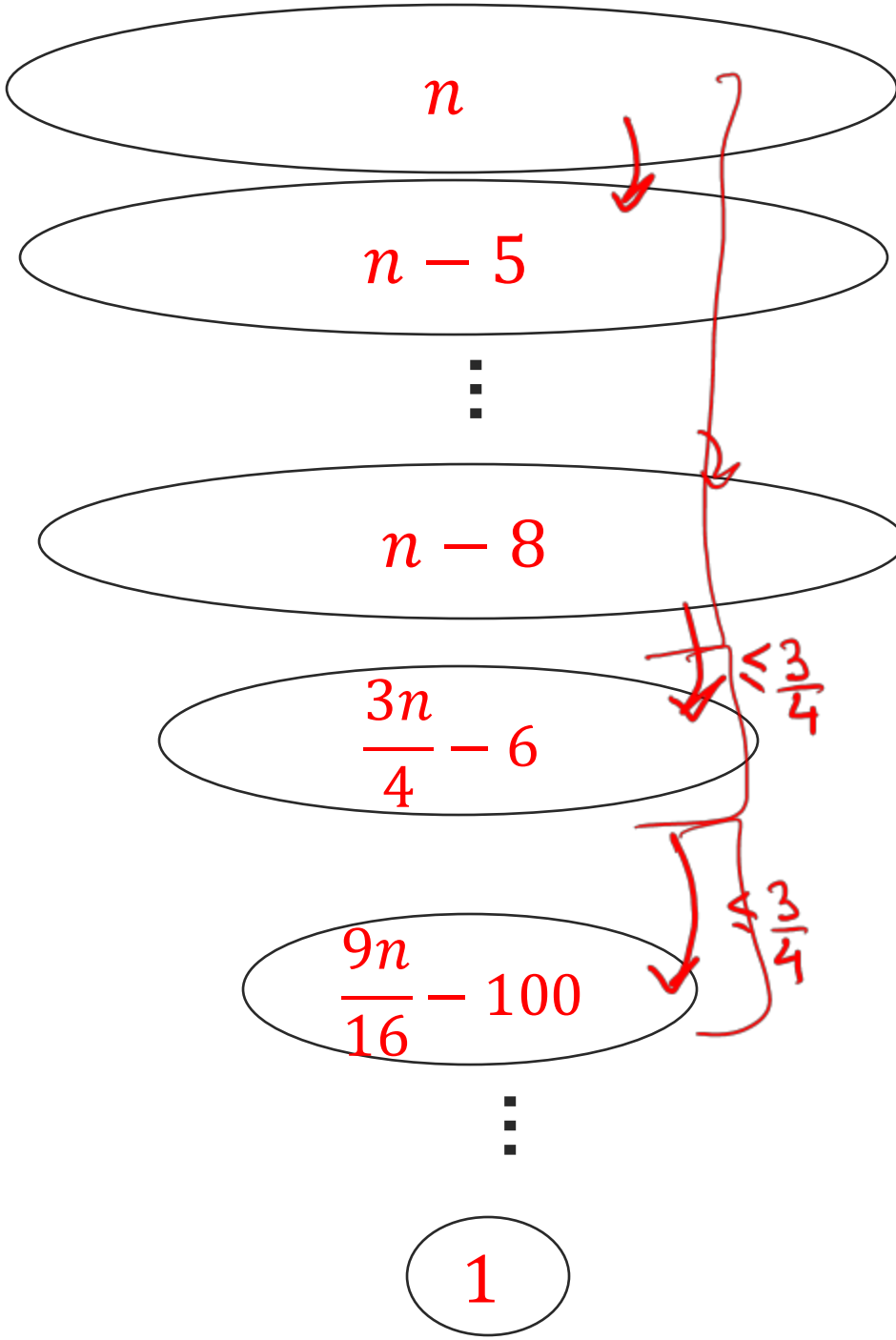
In reality, in some rounds we are using bad pivots and in some rounds we are using "good" pivots.

Whenever we get a "good" pivot, we multiply the problem size by $\leq 3/4$.

In some steps, we don't get a good pivot.

Divide the tree method to phases, indicating when the problem size shrinks by 3/4 .





Trees Revisited

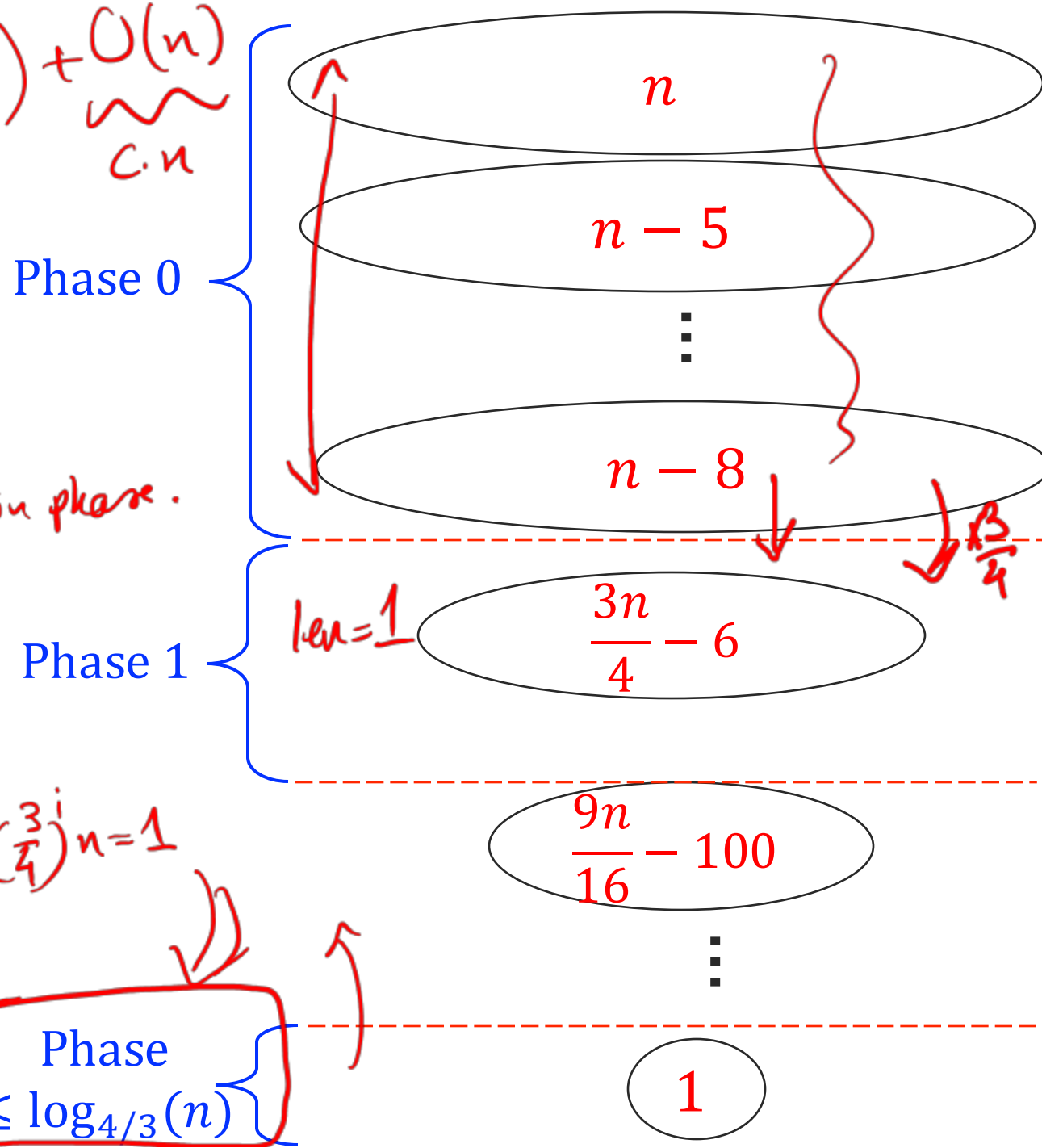
$T(n) \le T\left(\frac{3}{4}n\right) + O(n)$

$O(n) \sim c \cdot n$

Partition layers to "phases":

- Phase i include layers $[s_i, s_{i+1})$.
- s_{i+1}: layer when the problem size first becomes $\le \frac{3}{4}$ problem size of layer s_i.

$len(phase_i) = \#$ node in phase.



Lemma

In phase i, problem size $\le \left(\frac{3}{4}\right)^i n$

Lemma

Total runtime:

$$T(n) \le \sum_{i=0}^{\log_{4/3}(n)} len(Phase_i)\, c\, n \left(\frac{3}{4}\right)^i$$

Length of a Phase

The length of $Phase_i$:

- Random variable
- Eqv. how many tries it gets to shrink the problem size to ¾
- Length of phase ≤ # random pivots until we get a "good" pivot in phase i.

Lemma

Total runtime:

$$T(n) \leq \sum_{i=0}^{\log_{4/3}(n)} len(Phase_i)\, c\, n \left(\frac{3}{4}\right)^i$$

Random variable

Discuss

Every time we choose a pivot at random, with probability 50%, it is a ``good''. So, $\Pr[len(Phase_i) = 1] = 0.5$

What is $\Pr[len(Phase_i) = 2]$? $1/2^2$

What is $\Pr[len(Phase_i) = 3]$? $1/2^3$

What is $\Pr[len(Phase_i) = m]$? $1/2^m$

$\Pr[$1st 2nd ... $m-1$ bad pivot m^{th} good$] = (1/2)^m$

For ease, assume no element is repeated.
Remove this assumption at home!

Expected Phase Length

We want to compute the expected phase length $len(Phase_i)$

$$\mathbb{E}[len(Phase_i)] = \sum_{m=1}^{\infty} m \Pr[len(Phase_i) = m]$$

Discuss

Compute $\mathbb{E}[len(Phase_i)]$?

$\leq \sum_{m=0}^{\infty} m \left(\frac{1}{2}\right)^m = S$

$\frac{S}{2} = \sum_{m=0}^{\infty} m \left(\frac{1}{2}\right)^{m+1} = \sum_{m=0}^{\infty} (m+1)\left(\frac{1}{2}\right)^{m+1} - \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m+1}$

$= \sum_{m=1}^{\infty} m \left(\frac{1}{2}\right)^m = S$

$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^\infty} = 1$

$\Rightarrow \frac{S}{2} = S - 1 \Rightarrow S = 2$

Computing the Expected Runtime

Total expected runtime

$$\mathbb{E}[T(n)] \leq \mathbb{E}\left[\sum_{i=0}^{\log_{4/3}(n)} len(Phase_i)\; c\; n \left(\frac{3}{4}\right)^i\right]$$

$$= c\; n \sum_{i=0}^{\log_{4/3}(n)} \mathbb{E}[len(Phase_i)] \left(\frac{3}{4}\right)^i$$

$$\leq c\; n \sum_{i=0}^{\log_{4/3}(n)} 2 \left(\frac{3}{4}\right)^i \quad \in O(n)$$

Recall

$$T(n) \leq \sum_{i=0}^{\log_{4/3}(n)} len(Phase_i)\; c\; n \left(\frac{3}{4}\right)^i$$

Recall

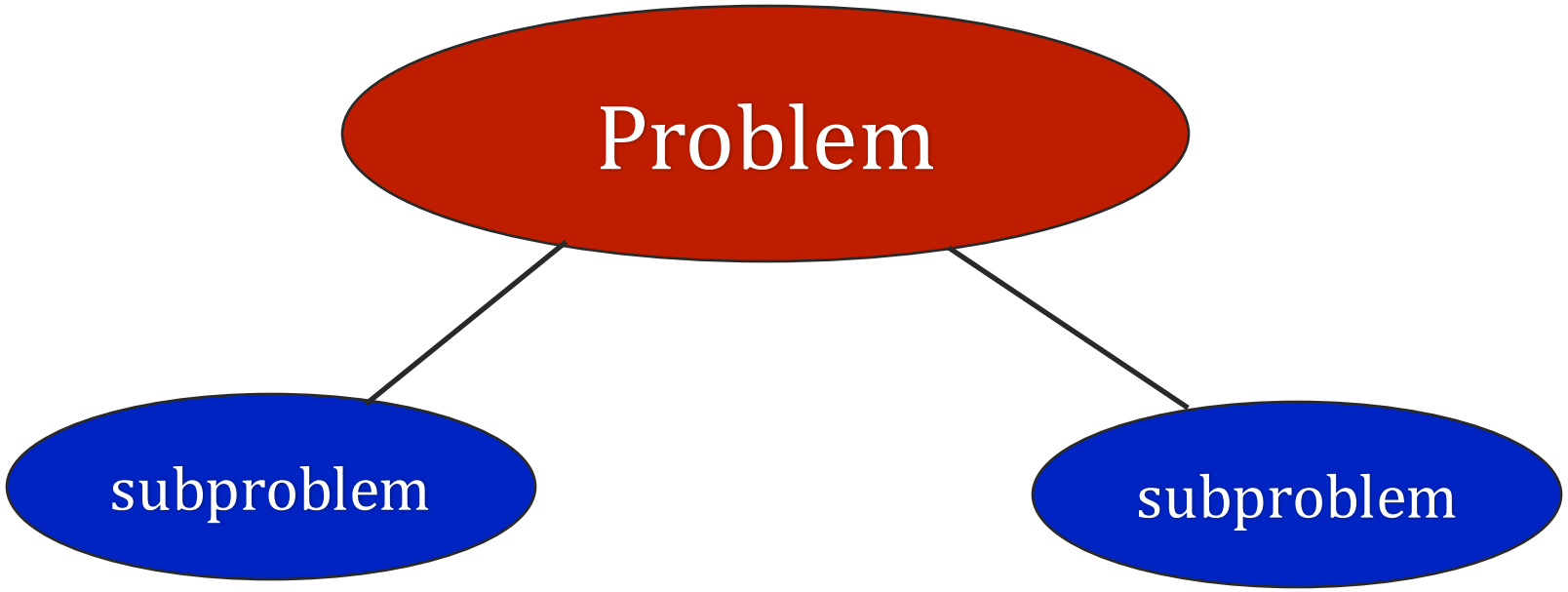
$$\mathbb{E}\left[len(Phase_i)\right] \leq 2$$

More on SELECT(S, k)

We gave a randomized algorithm, with *expected* $O(n)$ runtime.

There is also a cool deterministic algorithm, whose runtime is always $O(n)$.
→ It requires a more involved pivot selection mechanism
→ We won't talk about it in class.

The Big Picture of Divide and Conquer



First divide the problem to subproblems
→Subproblems are solved recursively

<u>Non-trivial Divide Step:</u>
QuickSelect, Strassen's, Karatsuba

Then combine the subproblems solutions to provide the answer to the big problem.

<u>Next: An example of non-trivial combine step</u>

ClosestPair

Input: Given n points on the plane $S = \{p_i = (x_i, y_i) \mid i = 1, \ldots, n\}$

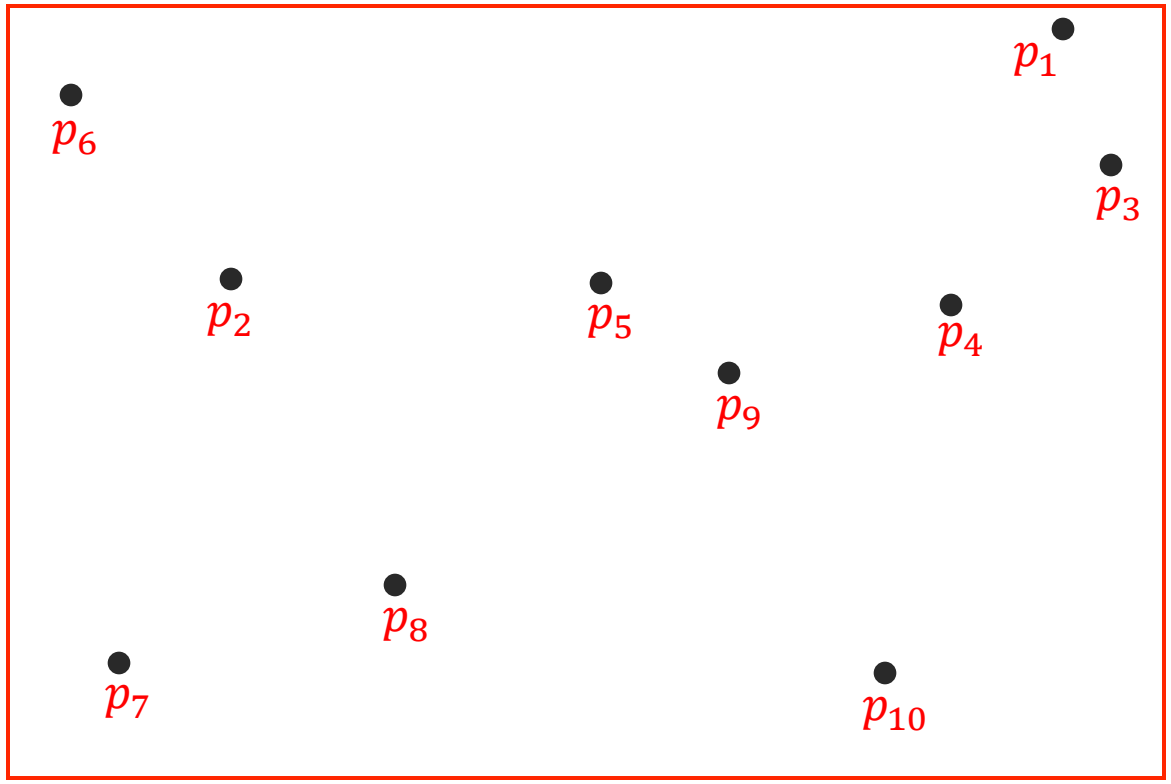
Output: Closest pair of points (p_i, p_j).

Naïve Algorithm:

- Compute all (p_i, p_j) distances, and output the pair with closest distance.
- $O(n^2)$ pairs, so $O(n^2)$ runtime.

Can we do better?

Can we get $O(n \ln(n))$?



Divide

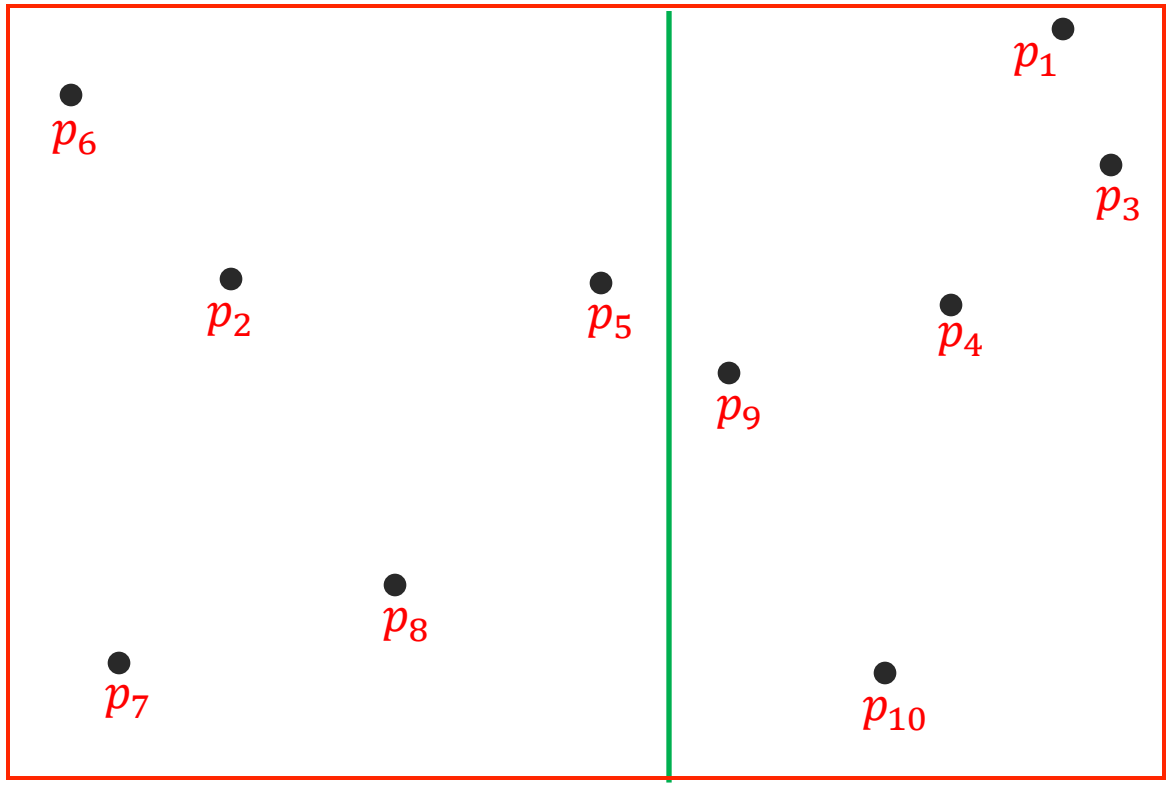
Since we are shooting for $O(n \ln(n))$, we might as well make two sorted lists according to the x-axis and y-axis at the beginning.

How should we divide?

- Divide on x-axis, along the *median*!
- Split the list (keep it sorted)

We have two subproblems

- $S_L = \{p_6, p_7, p_2, p_8, p_5\}$
- $S_R = \{p_9, p_{10}, p_4, p_1, p_3\}$



Where is the closest pair?

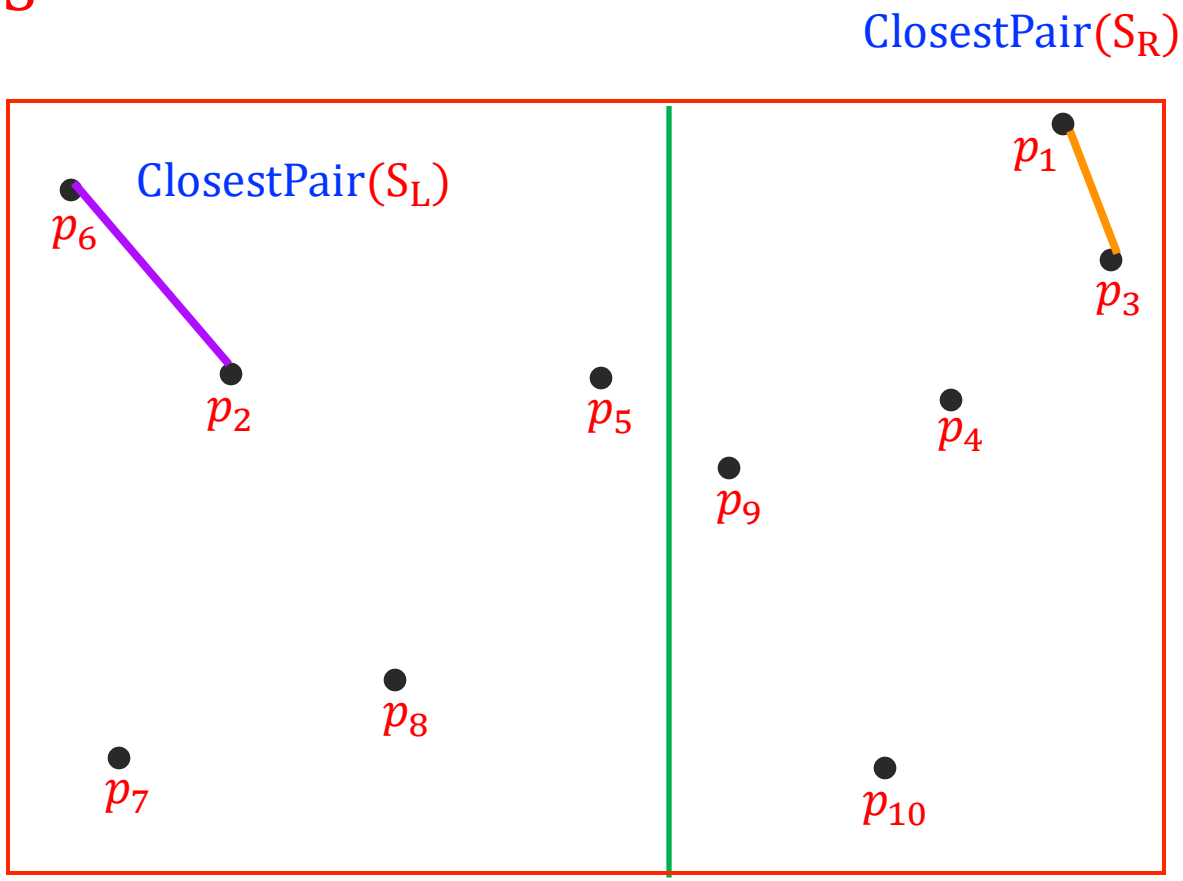
We should solve the two subproblems recursively:

- ClosestPair(S_L) = (p_6, p_2) and ClosestPair(S_R) = (p_1, p_3)
- One of these could be the closest pair in S
- Or, the closest pair crosses the median!

Checking for pairs across median naively:

→ Try every pair that crosses the median.

→ $\frac{n}{2} \times \frac{n}{2} \in \Theta(n^2)$



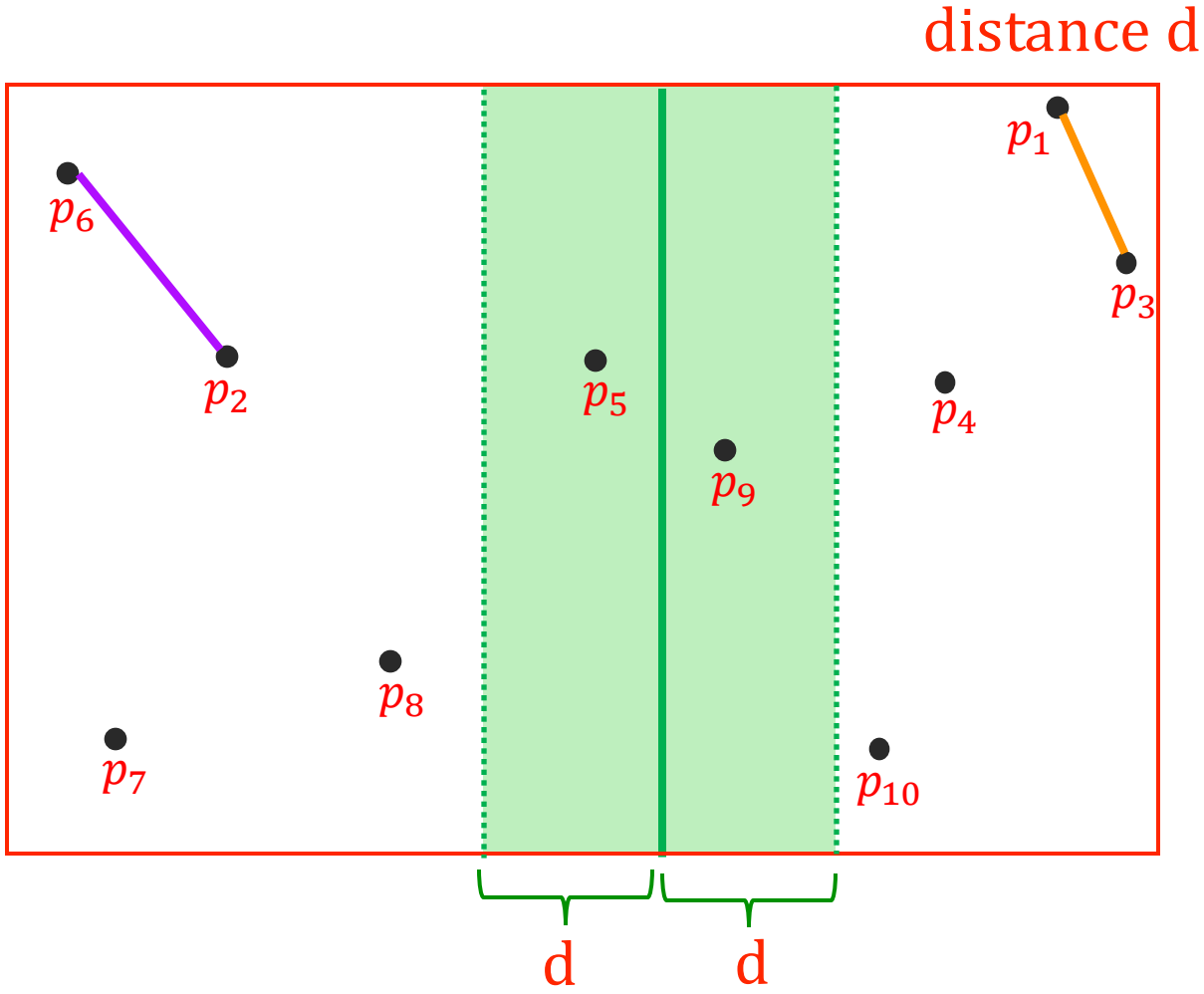
Idea 1

Let d be the min of two distances ClosestPair(S_L) and ClosestPair(S_R)

Then restrict attention to the strip of width d on each side of the median

This is not good enough

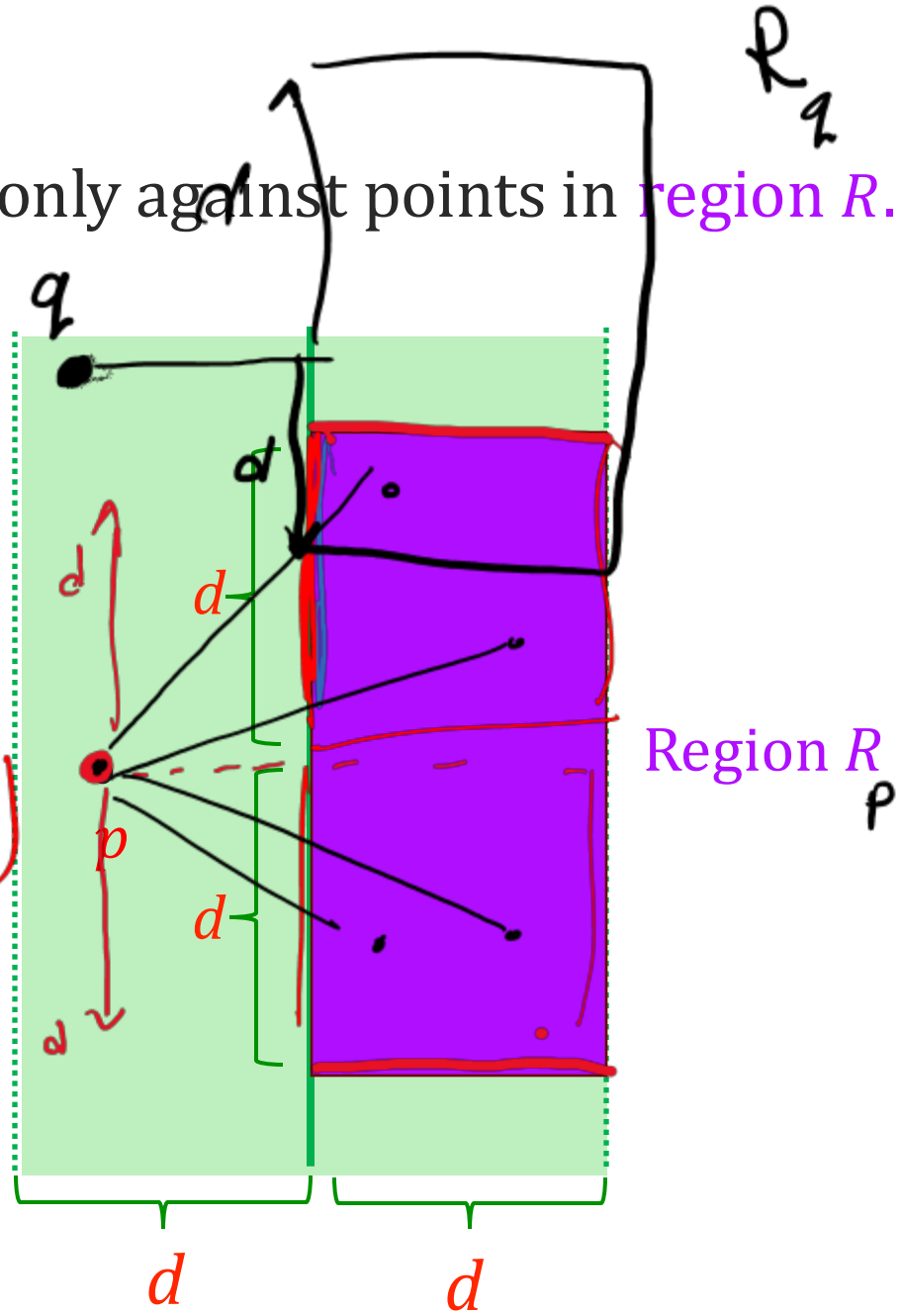
- In the worst-case, all points can be in this strip → Still take $\Theta(n^2)$.



Better Idea

Any point p in this strip, needs to be measured only against points in region R.

Region R: 2dxd on the other side of the strip from p.
- d higher to d lower
- Anything outside of R is either >d away from p. Or is on the same side as p (so would be caught by subproblems)



Claim

There are $O(1)$ points in R!

Divide and Conquer Algorithm

Let's believe in this claim that every p in strip needs to be measured against only $O(1)$ other points. What is the algorithm and runtime?

Write pseudo-code (esp. steps 4-5) Start with sorted arrays, both x and y

ClosestPair($\{p_i = (x_i, y_i) \mid i = 1, \dots, n\}$)

1. $(p_L, p'_L) \leftarrow$ ClosestPair$(p_1, \dots, p_{n/2})$
2. $(p_R, p'_R) \leftarrow$ ClosestPair$(p_{\frac{n}{2}+1}, \dots, p_n)$

$\}\ 2T\left(\frac{n}{2}\right)$

3. $d \leftarrow \min\{dist(p_L, p'_L), dist(p_R, p'_R)\}$
4. Compute points in the strip
5. For any point in the strip, compare the point to the $O(1)$ relevant points described in previous slide. Update minimum distance pair if needed.

$\}\ O(n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n) \in O(n \log(n))$$

Via Master Theorem

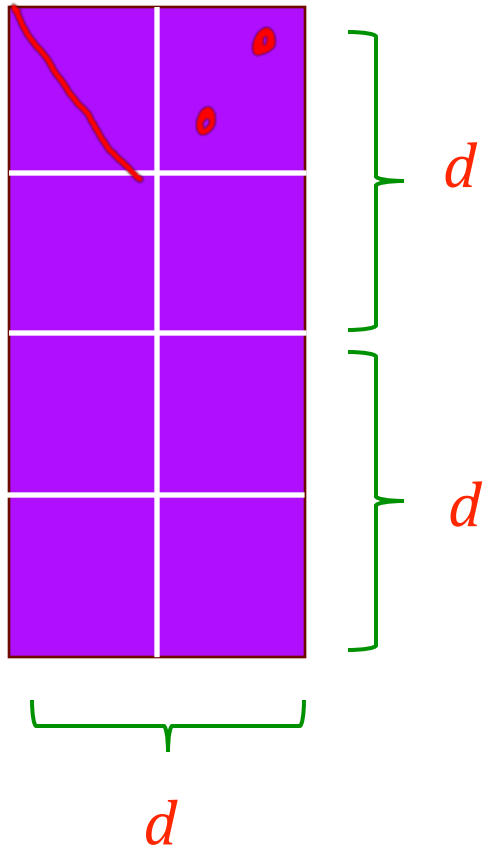
Proof that O(1) points are in Region R

$\sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{d}{2}\right)^2} \leq d$

Claim

There are ≤ 8 points in R!

Region R



Proof: Divide region R to 8 squares, if size $\frac{d}{2} \times \frac{d}{2}$.

Assume there are more than 8 points in R.

- Pigeon hole: One square has more than 1 point
- The diameter of these squares is less than d, so these points have distance $<$ d to each other
- They are both on the same side of the median too.

→ Contradicts that d was the defined as the smallest distance between a pair of points on either side of the median!

More on ClosestPair(S, k)

We gave a deterministic algorithm, with runtime $O(n \log(n))$.

There is also a cool randomized algorithm, whose expected runtime is $O(n)$.
→ It requires a more involved divide and conquer analysis
→ We won't talk about it in class.

Wrap up

In the first two weeks of class, we saw many examples of Divide and Conquer

- Integer and Matrix Multiplication
- (Median) Selection
- Closest Pair

- Solving recurrences
- Master theorem

Divide and Conquer
→ Is a powerful method
→ Both the divide and combine steps can be versatile

Next week:

Graph Problems and Algorithms

Prof. John Wright