Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Directed graphs...with cycles.
Quick Review:
DFS so far $\equiv$
Lecture in a minute!

Quick Review:
DFS so far ≡ how I learned to love the stack.
pre/post = time on stack.
Topological Ordering:
Inverse post ordering ≡ topological ordering.
Remove source, repeat.
Quick Review:
  DFS so far \(\equiv\) how I learned to love the stack.
  pre/post = time on stack.
Topological Ordering:
  Inverse post ordering \(\equiv\) topological ordering.
  Remove source, repeat.

Strongly Connected Components: directed graphs.
Strong Connectivity for \(u\) and \(v\).
  On a cycle together.
Easy: \(O(|V||E|)\) algorithm.
Linear time algorithm!
Lecture in a minute!

Quick Review:
  DFS so far $\equiv$ how I learned to love the stack.
  pre/post = time on stack.
Topological Ordering:
  Inverse post ordering $\equiv$ topological ordering.
  Remove source, repeat.

Strongly Connected Components: directed graphs.
Strong Connectivity for $u$ and $v$.
  On a cycle together.
Easy: $O(|V||E|)$ algorithm.
Linear time algorithm!
  Observation: Highest post in “source component”.
Find vertex in sink component.
  Explore.
Repeat.
DFS: so far.
DFS: so far.

How I learned to stop worrying
DFS: so far.

How I learned to stop worrying
...and love the stack.
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

[9x252]

See “Dr. Strangelove”.

Who is an ancestor of whom?

v is an ancestor of u

No cycles: no back edge

Back edge – edge to ancestor in “tree” of explore calls.

Topological order - edge (u, v) means u before v

Inverse post ordering is topological order.

Bonus Topological Order Algorithm:
remove source, repeat.
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([pre(u), post(u)]\) is “clock interval on stack.”
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

$[\text{pre}(u), \text{post}(u)]$ is “clock interval on stack.”

Who is an ancestor of whom?

\begin{center}
\begin{tabular}{c|c}
\cdots & \cdots \\
\hline
u & \ \ \ \ Who is an ancestor of whom? \\
\hline
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\end{tabular}
\end{center}
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([\text{pre}(u), \text{post}(u)]\) is “clock interval on stack.”

Who is an ancestor of whom?
\(v\) is an ancestor of \(u\)
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\[\text{[pre}(u), \text{post}(u)]\text{ is “clock interval on stack.”}\]

\[
\begin{array}{c}
\vdots \\
u \\
\vdots \\
v \\
\vdots \\
\end{array}
\]

Who is an ancestor of whom?

\(v\) is an ancestor of \(u\)

No cycles: no back edge
DFS: so far.

How I learned to stop worrying  
...and love the stack. See “Dr. Strangelove”.

[\text{pre}(u), \text{post}(u)] \text{ is “clock interval on stack.”}

\begin{array}{c}
  \vdots \\
  u \\
  \vdots \\
  v \\
  \vdots \\
\end{array}

Who is an ancestor of whom?

\( v \) is an ancestor of \( u \)

No cycles: no back edge

Back edge – edge to ancestor in “tree” of explore calls.”
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([pre(u) \text{, } post(u)]\) is “clock interval on stack.”

Who is an ancestor of whom?
\(v \text{ is an ancestor of } u\)

No cycles: no back edge
Back edge – edge to ancestor in “tree” of explore calls.”
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

$[pre(u), post(u)]$ is “clock interval on stack.”

...Who is an ancestor of whom?
$v$ is an ancestor of $u$

No cycles: no back edge
Back edge – edge to ancestor in “tree” of explore calls.”

Topological order - edge $(u, v)$ means $u$ before $v$
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([pre(u), post(u)]\) is “clock interval on stack.”

Who is an ancestor of whom?
\(v\) is an ancestor of \(u\)

No cycles: no back edge
Back edge – edge to ancestor in “tree” of explore calls.”

Topological order - edge \((u, v)\) means \(u\) before \(v\)
Inverse post ordering is topological order.
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([pre(u), post(u)]\) is “clock interval on stack.”

Who is an ancestor of whom?
\(v\) is an ancestor of \(u\)

No cycles: no back edge
Back edge – edge to ancestor in “tree” of explore calls.”

Topological order - edge \((u, v)\) means \(u\) before \(v\)
Inverse post ordering is topological order.
Bonus Topological Order Algorithm:
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([\text{pre}(u), \text{post}(u)]\) is “clock interval on stack.”

\[
\vdots
\quad u
\quad \vdots
\quad \vdots
\quad v
\quad \vdots
\quad \vdots
\]

Who is an ancestor of whom?
\(v\) is an ancestor of \(u\)

No cycles: no back edge
Back edge – edge to ancestor in “tree” of explore calls.

Topological order - edge \((u, v)\) means \(u\) before \(v\)
Inverse post ordering is topological order.

Bonus Topological Order Algorithm:
remove source,
DFS: so far.

How I learned to stop worrying
...and love the stack. See “Dr. Strangelove”.

\([pre(u), post(u)]\) is “clock interval on stack.”

\[
\begin{array}{c}
\vdots \\
u \\
\vdots \\
v \\
\vdots
\end{array}
\]

Who is an ancestor of whom?
\(v\) is an ancestor of \(u\)

No cycles: no back edge
Back edge – edge to ancestor in “tree” of explore calls.”

Topological order - edge \((u, v)\) means \(u\) before \(v\)
Inverse post ordering is topological order.

Bonus Topological Order Algorithm:
remove source, repeat.
Depth first search: directed.

Tree/forward edge \((u, v)\): int\((v)\) = [\text{pre}(v), \text{post}(v)] \in \text{int}(u) = [\text{pre}(u), \text{post}(u)].

Forward edge \((A, F)\): [10, 11] in [0, 13] or [0, [10, 11], 13]

Back edge \((u, v)\): int\((v)\) contains int\((u)\).

\((C, B)\): [3, 4] in [1, 8] or [1, [3, 4], 8]

Cross edge \((u, v)\): int\((v)\) before int\((u)\).

\((F, D)\): [2, 5] before [10, 11]
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)] \in \text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward \((A, F)\): \([10,11] \in [0,13] \) or \([0, [10,11], 13]\)

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\). \((C, B)\): \([3,4] \in [1,8] \) or \([1, [3,4], 8]\)

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\). \((F, D)\): \([2,5]\) before \([10,11]\)
Depth first search: directed.

- Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).
- Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).
- Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).
- \((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).
- Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).
- \((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge $(u, v)$: $\text{int}(v) = \text{[pre}(v), \text{post}(v)]$ in $\text{int}(u) = \text{[pre}(u), \text{post}(u)]$.

- Forward $(A, F)$: $\text{[10, 11]}$ in $\text{[0, 13]}$ or $\text{[0, [10, 11], 13]}$.
- Back edge $(C, B)$: $\text{int}(C)$ contains $\text{int}(B)$.
- Cross edge $(F, D)$: $\text{int}(F)$ before $\text{int}(D)$. 

Note: The diagram illustrates the directed graph with nodes labeled A to G.
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)] \in \text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10,11], 13]\).

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \([\text{pre}(v), \text{post}(v)]\) in \([\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).

Back edge \((u, v)\): \([\text{int}(v)]\) contains \([\text{int}(u)]\).

Cross edge \((u, v)\): \([\text{int}(v)]\) before \([\text{int}(u)]\).

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): int\((v)\) = [pre\((v)\), post\((v)\)] in int\((u)\) = [pre\((u)\), post\((u)\)].

Forward edge \((A, F)\): [10,11] in [0,13] or [0,[10,11],13]

Back edge \((u, v)\): int\((v)\) contains int\((u)\).

\((C, B)\): [3,4] in [1,8] or [1, [3, 4], 8]

Cross edge \((u, v)\): int\((v)\) before int\((u)\).

\((F, D)\): [2,5] before [10,11]
Depth first search: directed.

Tree/forward edge \((u, v)\): \([\text{pre}(v), \text{post}(v)]\) in \([\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).

Back edge \((u, v)\): \([\text{int}(v)]\) contains \([\text{int}(u)]\).

Cross edge \((u, v)\): \([\text{int}(v)]\) before \([\text{int}(u)]\).

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

**Tree/forward edge** \((u, v)\): int\((v)\) = [pre\((v)\), post\((v)\)] in int\((u)\) = [pre\((u)\), post\((u)\)].

**Forward** \((A, F)\): \([10,11]\) in \([0,13]\) or \([0, [10,11], 13]\)

**Back edge** \((u, v)\): int\((v)\) contains int\((u)\).

**Cross edge** \((u, v)\): int\((v)\) before int\((u)\).

\((F, D)\): \([2,5]\) before \([10,11]\)
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = \text{int}(u) = \text{int}(u, v)\) in \(\text{int}(u, v)\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\)

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).

\((F, D)\): \([2, 5]\) before \([10, 11]\)
Depth first search: directed.

Tree/forward edge \((u, v)\): \([\text{pre}(v), \text{post}(v)]\) in \([\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).

Back edge \((u, v)\): \([\text{int}(v)]\) contains \([\text{int}(u)]\).

Cross edge \((u, v)\): \([\text{int}(v)]\) before \([\text{int}(u)]\).

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge \( (u, v) \): \( \text{int}(v) = [\text{pre}(v), \text{post}(v)] \) in \( \text{int}(u) = [\text{pre}(u), \text{post}(u)] \).

Forward edge \( (A, F) \): \( [10, 11] \) in \( [0, 13] \) or \( [0, [10, 11], 13] \).

Back edge \( (u, v) \): \( \text{int}(v) \) contains \( \text{int}(u) \).

Cross edge \( (u, v) \): \( \text{int}(v) \) before \( \text{int}(u) \).

Back edge \( (C, B) \): \( [3, 4] \) in \( [1, 8] \) or \( [1, [3, 4], 8] \).

Cross edge \( (F, D) \): \( [2, 5] \) before \( [10, 11] \).
Depth first search: directed.

Tree/forward edge: $(u, v)$: $\text{int}(v) = [\text{pre}(v), \text{post}(v)]$ in $\text{int}(u) = [\text{pre}(u), \text{post}(u)]$.

Forward edge: $(A, F)$: $[10, 11]$ in $[0, 13]$ or $[0, [10, 11], 13]$.

Back edge: $(u, v)$: $\text{int}(v)$ contains $\text{int}(u)$.

Cross edge: $(u, v)$: $\text{int}(v)$ before $\text{int}(u)$.

$(C, B)$: $[3, 4]$ in $[1, 8]$ or $[1, [3, 4], 8]$. 


Depth first search: directed.

Tree/forward edge \((u, v)\): int\((v)\) = [\(\text{pre}(v), \text{post}(v)\)] in int\((u)\) = [\(\text{pre}(u), \text{post}(u)\)].

Forward \((A, F)\): [10, 11] in [0, 13] or [0, [10, 11], 13]

Back edge \((u, v)\): int\((v)\) contains int\((u)\).

\((C, B)\): [3, 4] in [1, 8] or [1, [3, 4], 8]

Cross edge \((u, v)\): int\((v)\) before int\((u)\).

\((F, D)\): [2, 5] before [10, 11]
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = \text{pre}(v), \text{post}(v)\) in \(\text{int}(u) = \text{pre}(u), \text{post}(u)\). 

Forward \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\). 

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\). 

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\). 

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\). 

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).

Back edge \((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = \text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward edge \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\).

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\).

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).

\((F, D)\): \([2, 5]\) before \([10, 11]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)] \) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)].\)

Forward \((A, F)\): [10,11] in [0,13] or [0,[10,11],13]
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\)

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward \((A, F)\): [10, 11] in [0, 13] or [0, [10, 11], 13]

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

\((C, B)\): [3, 4] in [1, 8] or [1, [3, 4], 8]
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v),\text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u),\text{post}(u)]\).

Forward \((A, F)\): \([10,11]\) in \([0,13]\) or \([0,[10,11],13]\)

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

\((C, B)\): \([3,4]\) in \([1,8]\) or \([1, [3,4], 8]\)

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).
Depth first search: directed.

Tree/forward edge \((u, v)\): \(\text{int}(v) = [\text{pre}(v), \text{post}(v)]\) in \(\text{int}(u) = [\text{pre}(u), \text{post}(u)]\).

Forward \((A, F)\): \([10, 11]\) in \([0, 13]\) or \([0, [10, 11], 13]\)

Back edge \((u, v)\): \(\text{int}(v)\) contains \(\text{int}(u)\).

\((C, B)\): \([3, 4]\) in \([1, 8]\) or \([1, [3, 4], 8]\)

Cross edge \((u, v)\): \(\text{int}(v)\) before \(\text{int}(u)\).

\((F, D)\): \([2, 5]\) before \([10, 11]\)
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:**
Testing for cycle/Topological sort.

Back Edge: \( (u, v) \) where \( int(v) \) contains \( int(u) \).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \( \implies \) cycle!
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!

   - Edge to ancestor, plus path from ancestor is cycle.
Testing for cycle/Topological sort.

Back Edge: $(u, v)$ where $int(v)$ contains $int(u)$.

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge $\implies$ cycle!
- Edge to ancestor, plus path from ancestor is cycle.
- Interval of first explored vertex, $v_0$, contains all others.
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
   Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
   Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
   - Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
   - Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
   - Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
   - Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.

Topological Ordering: \(\pi\), where for all edges \((u, v)\), \(\pi(u) < \pi(v)\).
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
   - Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
   - Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.

Topological Ordering: \(\pi\), where for all edges \((u, v)\), \(\pi(u) < \pi(v)\).

**Thm:** Reverse order of post number is a topological ordering.
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
  Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
  Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.

Topological Ordering: \(\pi\), where for all edges \((u, v)\), \(\pi(u) < \pi(v)\).

**Thm:** Reverse order of post number is a topological ordering.
Proof: No back edges!
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
   
   Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
   
   Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.

Topological Ordering: \(\pi\), where for all edges \((u, v)\), \(\pi(u) < \pi(v)\).

**Thm:** Reverse order of post number is a topological ordering.

Proof: No back edges!

   Tree/Forward edge: \((u, v)\) : \([\text{pre}(u), \text{post}(u)] \in [\text{pre}(v), \text{post}(v)]\)

   \(\implies\) \(\text{post}(u) > \text{post}(v)\).

   Cross edge: \((u, v)\) : \([\text{pre}(u), \text{post}(u)] > [\text{pre}(v), \text{post}(v)]\)

   \(\implies\) \(\text{post}(u) > \text{post}(v)\).
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!

Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.

Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.

Topological Ordering: \(\pi\), where for all edges \((u, v)\), \(\pi(u) < \pi(v)\).

**Thm:** Reverse order of post number is a topological ordering.

Proof: No back edges!

Tree/Forward edge: \((u, v)\) : \([\text{pre}(u), \text{post}(u)] \subseteq [\text{pre}(v), \text{post}(v)]\) \(\implies\) \(\text{post}(u) > \text{post}(v)\).

Cross edge: \((u, v)\) : \([\text{pre}(u), \text{post}(u)] > [\text{pre}(v), \text{post}(v)]\) \(\implies\) \(\text{post}(u) > \text{post}(v)\).

\(\implies\) for every edge, \((u, v)\), \(\text{post}(u) > \text{post}(v)\)
Testing for cycle/Topological sort.

Back Edge: \((u, v)\) where \(\text{int}(v)\) contains \(\text{int}(u)\).

**Thm:** A graph has a cycle if and only if there is back edge.

**Proof Idea:** Back edge \(\implies\) cycle!
   - Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, \(v_0\), contains all others.
   - Edge from “last” vertex, \(v_k\), to \(v_0\) is back edge.

Topological Ordering: \(\pi\), where for all edges \((u, v)\), \(\pi(u) < \pi(v)\).

**Thm:** Reverse order of post number is a topological ordering.

Proof: No back edges!
   - Tree/Forward edge: \((u, v) \colon [\text{pre}(u), \text{post}(u)] \in [\text{pre}(v), \text{post}(v)]\) \(\implies\) \(\text{post}(u) > \text{post}(v)\).
   - Cross edge: \((u, v) \colon [\text{pre}(u), \text{post}(u)] > [\text{pre}(v), \text{post}(v)]\) \(\implies\) \(\text{post}(u) > \text{post}(v)\).
   \[\implies\text{ for every edge, } (u, v), \text{post}(u) > \text{post}(v)\]
Topological Sort Example.

A linear order: A, E, F, B, G, D, C.

In DFS: When is A popped off stack?

Plus induction. \[ \Rightarrow \]

Reverse order of post numbering is topological ordering.
Topological Sort Example.

A linear order:

\[ A, E, F, B, G, D, C \]
Topological Sort Example.

A linear order:

\[ A, E, F, B, G, D, C \]
A linear order:

A, E, F, B, G, D, C

In DFS: When is A popped off stack?
A linear order:

\[A, E, F, B, G, D, C\]

In DFS: When is A popped off stack?

plus Induction.
Topological Sort Example.

A linear order:

$$A, E, F, B, G, D, C$$

In DFS: When is $A$ popped off stack?

plus Induction.  $\implies$ Reverse order of post numbering is topological ordering.
Topological Sort: DFS

Last post order should..

(A) be first in linearization!

(B) be last in linearization!
Topological Sort: DFS

Last post order should..

(A) be first in linearization!

(B) be last in linearization!

(A). First!
Connectivity in Directed Graphs.

Two nodes are connected...
Connectivity in Directed Graphs.

Two nodes are connected...when?

When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?
Both!

Nodes $u$ and $v$ are strongly connected if there is a path from $u$ to $v$ and a path from $v$ to $u$.

Note: Nodes are strongly connected to themselves. Path with zero edges in both directions!
Connectivity in Directed Graphs.

Two nodes are connected...when?
When there is a path from $u$ to $v$?
Connectivity in Directed Graphs.

Two nodes are connected...when?
When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?
Connectivity in Directed Graphs.

Two nodes are connected...when?
When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?
Both!
Connectivity in Directed Graphs.

Two nodes are connected...when?

When there is a path from \( u \) to \( v \)?
When there is a path from \( v \) to \( u \)?

Both!

Nodes \( u \) and \( v \) are **strongly connected**
Connectivity in Directed Graphs.

Two nodes are connected...when?

When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?

Both!

Nodes $u$ and $v$ are **strongly connected**
   if there is a path from $u$ to $v$
Connectivity in Directed Graphs.

Two nodes are connected...when?

When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?
Both!

Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$. 
Connectivity in Directed Graphs.

Two nodes are connected...when?
When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?
Both!

Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$.

Note: Nodes are strongly connected to themselves.
Connectivity in Directed Graphs.

Two nodes are connected...when?
When there is a path from $u$ to $v$?
When there is a path from $v$ to $u$?
Both!

Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$.

Note: Nodes are strongly connected to themselves.
Path with zero edges in both directions!
Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$. 

Remember: Nodes are strongly connected to themselves.

True/False?

If $u$ is strongly connected to $v$ and $v$ is strongly connected to $w$ = $\Rightarrow$ $u$ connected to $w$.

Transitive: $u$ strongly connected to $v$ strongly connected to $w$ = $\Rightarrow$ $u$ connected to $w$.

Relation = $\Rightarrow$ a partition into equivalence classes.

Strongly connected components: sets of nodes which are strongly connected.
Nodes $u$ and $v$ are **strongly connected** if there is a path from $u$ to $v$ and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.
Properties..

Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.

True/False?
Nodes \( u \) and \( v \) are **strongly connected**
  if there is a path from \( u \) to \( v \)
  and a path from \( v \) to \( u \).

Remember: Nodes are strongly connected to themselves.

True/False?

If \( u \) is strongly connected to \( v \) and \( v \) is strongly connected to \( w \)
\[ \implies u \text{ connected to } w. \]
Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.

True/False?
If $u$ is strongly connected to $v$ and $v$ is strongly connected to $w$
$\implies u$ connected to $w$.

True!
Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.

True/False?

If $u$ is strongly connected to $v$ and $v$ is strongly connected to $w$
$\implies u$ connected to $w$.

True!

path from $u$ (through $v$) to $w$ and path from $w$ (through $v$) to $u$!
Nodes \( u \) and \( v \) are **strongly connected**
if there is a path from \( u \) to \( v \)
and a path from \( v \) to \( u \).

Remember: Nodes are strongly connected to themselves.

True/False?

If \( u \) is strongly connected to \( v \) and \( v \) is strongly connected to \( w \)
\( \implies \) \( u \) connected to \( w \).

True!

path from \( u \) (through \( v \)) to \( w \) and path from \( w \) (through \( v \)) to \( u \)!

Transitive: \( u \) strongly connected to \( v \) strongly connected to \( w \)
Properties..

Nodes $u$ and $v$ are **strongly connected**
if there is a path from $u$ to $v$
and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.

True/False?

If $u$ is strongly connected to $v$ and $v$ is strongly connected to $w$
$\implies u$ connected to $w$.

True!

path from $u$ (through $v$) to $w$ and path from $w$ (through $v$) to $u$!

Transitive: $u$ strongly connected to $v$ strongly connected to $w$
$\implies u$ connected to $w$. 
Nodes $u$ and $v$ are **strongly connected**
   if there is a path from $u$ to $v$
   and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.

True/False?

If $u$ is strongly connected to $v$ and $v$ is strongly connected to $w$
$\implies u$ connected to $w$.

True!
   path from $u$ (through $v$) to $w$ and path from $w$ (through $v$) to $u$!

Transitive: $u$ strongly connected to $v$ strongly connected to $w$
$\implies u$ connected to $w$.

Relation $\implies$ a partition into equivalence classes.
Properties..

Nodes $u$ and $v$ are **strongly connected** if there is a path from $u$ to $v$ and a path from $v$ to $u$.

Remember: Nodes are strongly connected to themselves.

True/False?

If $u$ is strongly connected to $v$ and $v$ is strongly connected to $w$ \[\Rightarrow\] $u$ connected to $w$.

True!

path from $u$ (through $v$) to $w$ and path from $w$ (through $v$) to $u$!

Transitive: $u$ strongly connected to $v$ strongly connected to $w$ \[\Rightarrow\] $u$ connected to $w$.

Relation \[\Rightarrow\] a partition into equivalence classes.

**Strongly connected components:** sets of nodes which are strongly connected.
Example.

Collapsing strongly connected components (SCCs) yields a DAG! Why? Any cycle collapses nodes into a single SCC.
Collapsing strongly connected components (SCCs) yields a DAG!

Why? Any cycle collapses nodes into a single SCC.
Example.

Collapsing strongly connected components (SCCs)..

Why?

Any cycle collapses nodes into a single SCC.
Example.

Collapsing strongly connected components (SCCs) yields a DAG!

Why? Any cycle collapses nodes into a single SCC.
Example.

Collapsing strongly connected components (SCCs) yields a DAG!

Why?
Example.

Collapsing strongly connected components (SCCs)...
..yields a DAG!

Why?
..any cycle collapses nodes into a single SCC.
Dag of SCCs

**Property:** Every directed graph is a DAG of strongly connected components.
Dag of SCCs

**Property:** Every directed graph is a DAG of strongly connected components.

Finding the strongly connected components?
Dag of SCCs

**Property:** Every directed graph is a DAG of strongly connected components.

Finding the strongly connected components?

**Property:** \( \text{explore}(u) \) visits all nodes reachable from \( u \).
**Property:** Every directed graph is a DAG of strongly connected components.

**Algorithm:**
1. Run explore on node in sink component.
2. Output visited nodes.
3. Repeat.

How do we find a node in the sink component?

**Property:** \( \text{explore}(u) \) visits all nodes reachable from \( u \).
Property: Every directed graph is a DAG of strongly connected components.

Finding the strongly connected components?

Property: \text{explore}(u) \text{ visits all nodes reachable from } u.

Algorithm:
1. Run \text{explore} on node in sink component.
   
   get all nodes in sink.
Dag of SCCs

**Property:** Every directed graph is a DAG of strongly connected components.

- **Property:** explore($u$) visits all nodes reachable from $u$.

**Algorithm:**
1. Run explore on node in sink component. **get all nodes in sink.**
2. Output visited nodes.
**Dag of SCCs**

**Property:** Every directed graph is a DAG of strongly connected components.

Finding the strongly connected components?

**Property:** \(\text{explore}(u)\) visits all nodes reachable from \(u\).

**Algorithm:**
1. Run explore on node in sink component.
   
   get all nodes in sink.
2. Output visited nodes.
3. Repeat.
Property: Every directed graph is a DAG of strongly connected components.

Finding the strongly connected components?

Property: \text{explore}(u) \text{ visits all nodes reachable from } u.

Algorithm:
1. Run explore on node in sink component.
   \text{get all nodes in sink.}
2. Output visited nodes.
3. Repeat.

How do we find a node in the sink component?
Finding a source.

**Property**: The node with the highest post order number is in a source component.
Property: The node with the highest post order number is in a source component.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Post numbers of SCCs

**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Post numbers of SCCs

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.
Post numbers of SCCs

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.
Post numbers of SCCs

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Post numbers of SCCs

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.
Post numbers of SCCs

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Highest post# in $C$ bigger than any in $C'$

not true every post# in $C$ greater than any in $C'$
Proof:

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 

Proof:
Proof:

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Proof:

If a node $v$ in $C'$ is explored first.
Proof:

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Proof:

If a node $v$ in $C'$ is explored first.
   - $\text{explore}(v)$ gets to all of $C'$
   - and none of $C$!
So every node in $C$ explored after every node in $C'$. 
**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

**Proof:**

If a node $v$ in $C'$ is explored first.
- `explore(v)` gets to all of $C'$
- and none of $C$!

So every node in $C$ explored after every node in $C'$.
- $\equiv$ every post # in $C$ larger than every post # in $C'$. 
**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

**Proof:**

If a node $v$ in $C'$ is explored first.

- `explore(v)` gets to all of $C'$
- and none of $C$!

So every node in $C$ explored after every node in $C'$.

$\equiv$ every post # in $C$ larger than every post # in $C'$.

If a node $u$ in $C$ is explored first
**Proof:**

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

**Proof:**

If a node $v$ in $C'$ is explored first.

- `explore(v)` gets to all of $C'$
- and none of $C$!

So every node in $C$ explored after every node in $C'$.

\[ \equiv \] every post # in $C$ larger than every post # in $C'$.

If a node $u$ in $C$ is explored first

- All of $C$ and $C'$ will be explored before returning
Proof:

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Proof:

If a node $v$ in $C'$ is explored first.
  
  \[
  \text{explore}(v) \text{ gets to all of } C' \\
  \text{and none of } C!
  \]

So every node in $C$ explored after every node in $C'$.
  
  \[
  \equiv \text{ every post # in } C \text{ larger than every post # in } C'.
  \]

If a node $u$ in $C$ is explored first

All of $C$ and $C'$ will be explored before returning from \texttt{explore}(u)
Proof:

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Proof:

If a node $v$ in $C'$ is explored first.
   - `explore(v)` gets to all of $C'$
   - and none of $C$!
So every node in $C$ explored after every node in $C'$.
   - $\equiv$ every post # in $C$ larger than every post # in $C'$.

If a node $u$ in $C$ is explored first
   - All of $C$ and $C'$ will be explored before returning from `explore(u)`
So $u$ has higher post number than any node in $C'$. 
Proof:

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

**Proof:**

If a node $v$ in $C'$ is explored first.

- `explore(v)` gets to all of $C'$
- and none of $C$!

So every node in $C$ explored after every node in $C'$.

$\equiv$ every post # in $C$ larger than every post # in $C'$.

If a node $u$ in $C$ is explored first

All of $C$ and $C'$ will be explored before returning from `explore(u)`

So $u$ has higher post number than any node in $C'$. 

\[\square\]
**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

**Proof:**

If a node $v$ in $C'$ is explored first.
- $\text{explore}(v)$ gets to all of $C'$
- and none of $C$!

So every node in $C$ explored after every node in $C'$.
- $\equiv$ every post # in $C$ larger than every post # in $C'$.

If a node $u$ in $C$ is explored first
- All of $C$ and $C'$ will be explored before returning from $\text{explore}(u)$

So $u$ has higher post number than any node in $C'$.

Implies highest post numbered node is in source component.
Proof:

Property++: If C and C' are SCCs with an edge from C to C', highest post# of a node in C larger than post# of any node in C'.

Proof:

If a node v in C' is explored first.
   \[ \text{explore}(v) \text{ gets to all of } C' \]
   and none of C!
So every node in C explored after every node in C'.
   \[ \equiv \text{ every post # in } C \text{ larger than every post # in } C' \]

If a node u in C is explored first
   All of C and C' will be explored before returning from \textbf{explore}(u)
So u has higher post number than any node in C'.

Implies highest post numbered node is in source component.
Test your understanding..

**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes!
Test your understanding..

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily.
**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)
Test your understanding..

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

```
Explore(a) =⇒ Explore(b) =⇒ Explore(c) =⇒ Return from (c) =⇒ Explore(d) =⇒ Return from (d)
```

(c) has lower post order number than (d).
**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

![Diagram]

Explore(a)
**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

![Graph](image)

Explore(a) \[\Rightarrow\]
**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

```
Explore(a) ⇒ Explore(b)
```

![Diagram](image-url)
Test your understanding.

**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

```
Explore(a) ➔ Explore(b) ➔ Explore (c)
```

![Graph diagram](attachment:image.png)
**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$? 

(A) Yes! (B) Not Necessarily. ... (B)

```
Explore(a) ➞ Explore(b) ➞ Explore (c) ➞ Return from (c)
```
Test your understanding..

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes!  (B) Not Necessarily. ... (B)

![Graph](image)

Explore(a) $\implies$ Explore(b) $\implies$ Explore (c) $\implies$ Return from (c) $\implies$ Explore(d)
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

```
Explore(a) \implies Explore(b) \implies Explore (c) \\
\implies Return from (c) \implies Explore(d) \implies Return from (d) ...
```
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

Does every node in $C$ have a higher post order number than every node in $C'$?

(A) Yes! (B) Not Necessarily. ... (B)

Explore(a) $\implies$ Explore(b) $\implies$ Explore (c) $\implies$ Return from (c) $\implies$ Explore(d) $\implies$ Return from (d) ...

(c) has lower post order number than (d).
Property: The highest post numbered node is in source component.
**Property:** The highest post numbered node is in source component.

Algorithm:
1. Run explore on node in sink component.
2. Output visited nodes.
3. Repeat.
Property: The highest post numbered node is in source component.

Algorithm:
1. Run explore on node in sink component.
2. Output visited nodes.
3. Repeat.

Uh...oh.
Property: The highest post numbered node is in source component.

Algorithm:
1. Run explore on node in sink component.
2. Output visited nodes.
3. Repeat.

Uh...oh.

How should we fix this?
SCC Algorithm.

**Property:** The highest post numbered node is in source component.
SCC Algorithm.

**Property:** The highest post numbered node is in source component.
Find node in sink component?

1. DFS on $G^R$ to compute post $(·)$
   Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.
2. Output nodes visited in: explore($v$)
Then what?
Find another node in sink of unvisited part of $G$!
Recompute DFS in $G^R$...
or...
.... use post $(·)$ again!
SCC Algorithm.

**Property:** The highest post numbered node is in **source** component.

Find node in sink component?

Reverse edges!
SCC Algorithm.

Property: The highest post numbered node is in source component. Find node in sink component?

Reverse edges! $G^R$
SCC Algorithm.

**Property:** The highest post numbered node is in source component. Find node in sink component? Reverse edges! $G^R$

Source component in $G^R$ is sink component in $G$. 
Property: The highest post numbered node is in source component.

Find node in sink component?

Reverse edges! $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:
Property: The highest post numbered node is in source component.

Find node in sink component?

Reverse edges! $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
**SCC Algorithm.**

**Property:** The highest post numbered node is in source component.

Find node in sink component?

Reverse edges! $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:

1. DFS on $G^R$ to compute \(\text{post}(\cdot)\)

   Highest post # vertex, \(v\), in $G^R$ in sink comp. of $G$. 
SCC Algorithm.

**Property:** The highest post numbered node is in source component. Find node in sink component?

**Reverse edges!** $G^R$

Source component in $G^R$ is sink component in $G$.

**Algorithm:**

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
   
   Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.

2. Output nodes visited in: $\text{explore}(v)$
**SCC Algorithm.**

**Property:** The highest post numbered node is in source component.

Find node in sink component?

Reverse edges! $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
   - Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.
2. Output nodes visited in: $\text{explore}(v)$

Then what?
SCC Algorithm.

**Property:** The highest post numbered node is in source component.

Find node in sink component?

Reverse edges! $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
   
   Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.

2. Output nodes visited in: $\text{explore}(v)$
   
   Then what?

Find another node in sink of unvisited part of $G$!
SCC Algorithm.

**Property:** The highest post numbered node is in source component.

Find node in sink component?

**Reverse edges!** $G^R$

Source component in $G^R$ is sink component in $G$.

**Algorithm:**

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
   
   Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.

2. Output nodes visited in: $\text{explore}(v)$
   
   Then what?

Find another node in sink of unvisited part of $G$!

Recompute DFS in $G^R$...
SCC Algorithm.

**Property:** The highest post numbered node is in source component.

Find node in sink component?

**Reverse edges!** $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
   - Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.
2. Output nodes visited in: $\text{explore}(v)$
   - Then what?

Find another node in sink of unvisited part of $G$!

Recompute DFS in $G^R$...or...
SCC Algorithm.

**Property:** The highest post numbered node is in source component.

Find node in sink component?

**Reverse edges!** $G^R$

Source component in $G^R$ is sink component in $G$.

Algorithm:

1. DFS on $G^R$ to compute $\text{post}(\cdot)$
   Highest post # vertex, $v$, in $G^R$ in sink comp. of $G$.
2. Output nodes visited in: $\text{explore}(v)$
   Then what?

Find another node in sink of unvisited part of $G$!

Recompute DFS in $G^R$...or...

.... use $\text{post}(\cdot)$ again!
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, the highest post# of a node in $C$ is larger than the post# of any node in $C'$. 
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Property++: If \( C \) and \( C' \) are SCCs with an edge from \( C \) to \( C' \), the highest post# of a node in \( C \) is larger than the post# of any node in \( C' \).
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, the highest post# of a node in $C$ is larger than the post# of any node in $C'$.
**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Post from Reverse graph

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 

First explore of $G$: 

\[
\text{Removes sink component of } G \Rightarrow \text{removes source component of } G_R.
\]

\[
\text{highest rem. post # vertex, } v, \text{ in } G_R \text{ in component with no in-edges} \Rightarrow v \text{ in source component of } G_R \Rightarrow v \text{ in sink component of } G.
\]
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 

First explore of $G$:
Removes sink component of $G$. 

Compute $G_R$ in linear time?...
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
\[\Rightarrow\] removes source component of $G^R$. 
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
$\implies$ removes source component of $G^R$.
$\implies$ highest rem. post # vertex, $v$,
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
$\implies$ removes source component of $G^R$.
$\implies$ highest rem. post # vertex, $v$,
in $G^R$ in component with no in-edges
All sinks from one dfs.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
$\implies$ removes source component of $G^R$.
$\implies$ highest rem. post # vertex, $v$, in $G^R$ in component with no in-edges
$\implies$ in source component of $G^R$
All sinks from one dfs.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
  $\implies$ removes source component of $G^R$.
  $\implies$ highest rem. post # vertex, $v$,
in $G^R$ in component with no in-edges
  $\implies$ in source component of $G^R$
  $\implies$ $v$ in sink component of $G$!
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
$\implies$ removes source component of $G^R$.
$\implies$ highest rem. post # vertex, $v$,
in $G^R$ in component with no in-edges
$\implies$ in source component of $G^R$
$\implies$ $v$ in sink component of $G$!

SCC Algorithm:
All sinks from one dfs.

**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
\[\Rightarrow\] removes source component of $G^R$.
\[\Rightarrow\] highest rem. post # vertex, $v$, in $G^R$ in component with no in-edges
\[\Rightarrow\] in source component of $G^R$
\[\Rightarrow\] $v$ in sink component of $G$!

SCC Algorithm:
1. DFS of $G^R$. 

All sinks from one dfs.

**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
\[ \Rightarrow \] removes source component of $G^R$.
\[ \Rightarrow \] highest rem. post # vertex, $v$, in $G^R$ in component with no in-edges
\[ \Rightarrow \] in source component of $G^R$
\[ \Rightarrow \] $v$ in sink component of $G$!

SCC Algorithm:
1. DFS of $G^R$.
2. Run undirected components algorithm on $G$. 

Compute $G^R$ in linear time?.. exercise.
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.

$\implies$ removes source component of $G^R$.
$\implies$ highest rem. post # vertex, $v$, in $G^R$ in component with no in-edges
$\implies$ in source component of $G^R$
$\implies$ $v$ in sink component of $G$!

---

**SCC Algorithm:**
1. DFS of $G^R$.
2. Run undirected components algorithm on $G$ — in reverse post order number from step 1.
All sinks from one dfs.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
\[ \implies \text{removes source component of } G^R. \]
\[ \implies \text{highest rem. post # vertex, } v, \text{ in } G^R \text{ in component with no in-edges} \]
\[ \implies \text{in source component of } G^R \]
\[ \implies v \text{ in sink component of } G! \]

SCC Algorithm:
1. DFS of $G^R$.
2. Run undirected components algorithm on $G$ — in reverse post order number from step 1.

$O(|V| + |E|)$ time ...
All sinks from one dfs.

**Property++:** If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
\[\implies\] removes source component of $G^R$.
\[\implies\] highest rem. post # vertex, $v$,
in $G^R$ in component with no in-edges
\[\implies\] in source component of $G^R$
\[\implies\] $v$ in sink component of $G$!

SCC Algorithm:
1. DFS of $G^R$.
2. Run undirected components algorithm on $G$
   — in reverse post order number from step 1.

$O(|V| + |E|)$ time ...

Compute $G^R$ in linear time?..
All sinks from one dfs.

**Property++**: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$.

First explore of $G$:
Removes sink component of $G$.
$\implies$ removes source component of $G^R$.
$\implies$ highest rem. post # vertex, $v$, in $G^R$ in component with no in-edges
$\implies$ in source component of $G^R$
$\implies$ $v$ in sink component of $G$!

---

SCC Algorithm:
1. DFS of $G^R$.
2. Run undirected components algorithm on $G$ — in reverse post order number from step 1.

$O(|V| + |E|)$ time ...

Compute $G^R$ in linear time?.. exercise.
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, the highest post# of a node in $C$ is larger than the post# of any node in $C'$. 
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, the highest post# of a node in $C$ is larger than the post# of any node in $C'$. 
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, the highest post# of a node in $C$ is larger than the post# of any node in $C'$. 
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Example Again: think runtime.

Property++: If $C$ and $C'$ are SCCs with an edge from $C$ to $C'$, highest post# of a node in $C$ larger than post# of any node in $C'$. 
Lecture in a minute!

Quick Review:
DFS so far $\equiv$

Topological Ordering:
Inverse post ordering $\equiv$ topological ordering.
Remove source, repeat.

Strongly Connected Components: directed graphs.
Strong Connectivity for $u$ and $v$.
On a cycle together.
Easy: $O(|V||E|)$ algorithm.
Linear time algorithm!
Observation: Highest post in "source component".
Find vertex in sink component.
Explore.
Repeat.
Quick Review:
   DFS so far ≡ how I learned to love the stack.
   pre/post = time on stack.
Topological Ordering:
   Inverse post ordering ≡ topological ordering.
   Remove source, repeat.
Quick Review:
   DFS so far $\equiv$ how I learned to love the stack.
   pre/post = time on stack.
Topological Ordering:
   Inverse post ordering $\equiv$ topological ordering.
   Remove source, repeat.

Strongly Connected Components: directed graphs.
Strong Connectivity for $u$ and $v$.
   On a cycle together.
Easy: $O(|V||E|)$ algorithm.
Linear time algorithm!
Lecture in a minute!

Quick Review:
- DFS so far \( \equiv \) how I learned to love the stack.
  - pre/post = time on stack.
- Topological Ordering:
  - Inverse post ordering \( \equiv \) topological ordering.
  - Remove source, repeat.

Strongly Connected Components: directed graphs.
- Strong Connectivity for \( u \) and \( v \).
  - On a cycle together.
- Easy: \( O(|V||E|) \) algorithm.
- Linear time algorithm!
  - Observation: Highest post in “source component”.
  - Find vertex in sink component.
  - Explore.
  - Repeat.