CS 170
Efficient Algorithms and Intractable Problems

Lecture 8
Greedy Algorithms

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Announcements

Midterm 1 on Feb 25.
→ Scope: everything up to and including Feb 20 lecture.
→ See Ed for past midterm megathreads
→ Stay tuned for logistics post next week

Homeworks:
→HW 3 is due this Saturday
→ HW 4 will be posted on Sunday

Anonymous feedback form is set up
→ Access through Edstem logistics thread.

Last two lectures

Lots of graph algorithms

- BFS, DFS
- Applications of BFS, DFS

Today and Next 2 Lecture: Greedy Algorithms

Algorithms that build up a solution

piece by piece, always choosing the next piece

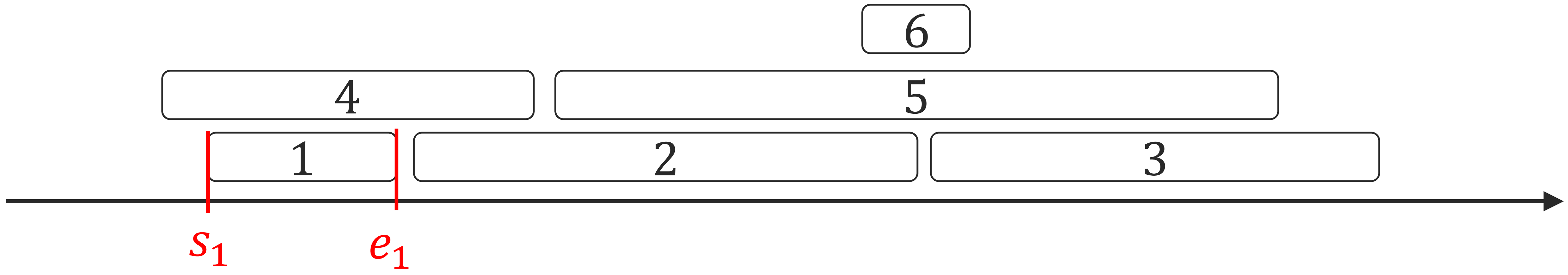
that offers the most obvious and immediate benefit!

Examples of problems where greedy works
- Scheduling
- Satisfiability
- Huffman Coding (next lecture probably)
- Minimum spanning trees (next lecture)

(Interval) Scheduling

Input: collection of n jobs specified by their time intervals $[s_1, e_1], \dots, [s_n, e_n]$.

Goal: Find the largest subset of jobs, that have no time conflicts.



Application example:

- intervals denote activities you are interested in.

→ classes take, times you can hangout with friends, time for rest, appointments, …

- You want to do as many activities as possible!

(Interval) Scheduling

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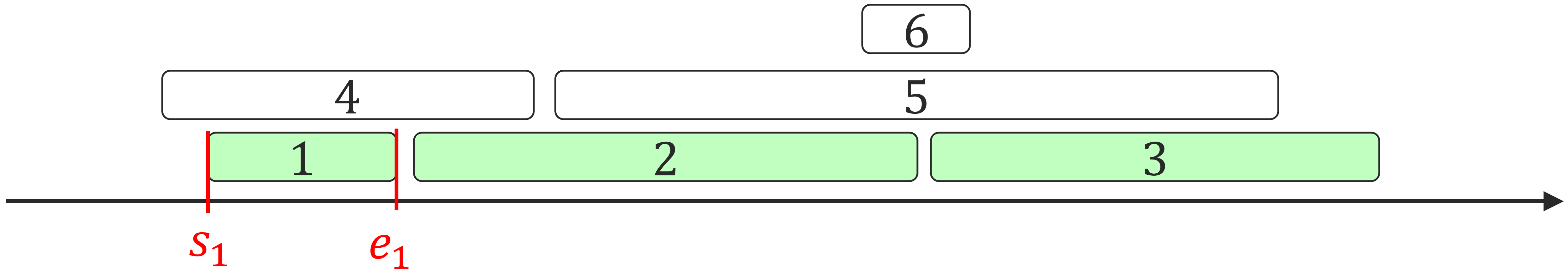
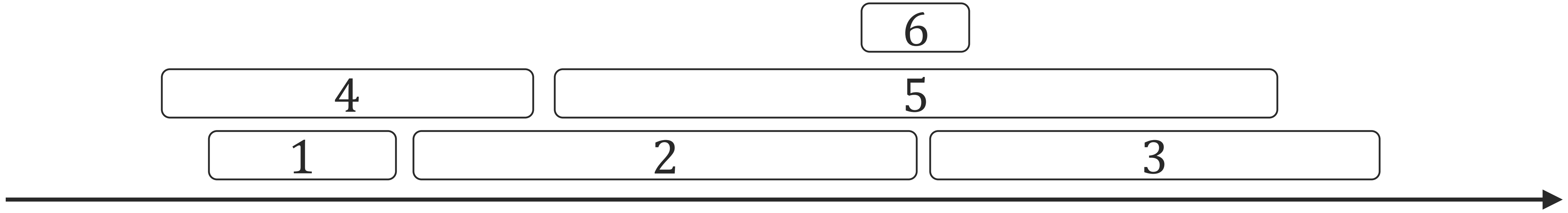


Discuss

Let's pick greedily! Which interval should we pick next?

- Shortest job?
- Earliest start time?
- Earliest end time?

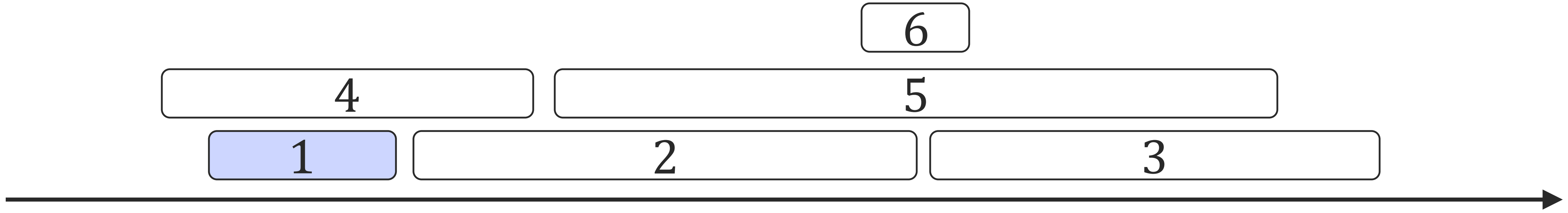
Pick the earliest finish time, and repeat!



Algorithm:

```
While the set of intervals is non-empty
    Add interval $j$ with the earliest finish time $e_j$.
    Remove any conflicted interval $i$ from the set, i.e., $[s_j, e_j] \cap [s_i, e_i] \neq \emptyset$
```

Pick the earliest finish time, and repeat!



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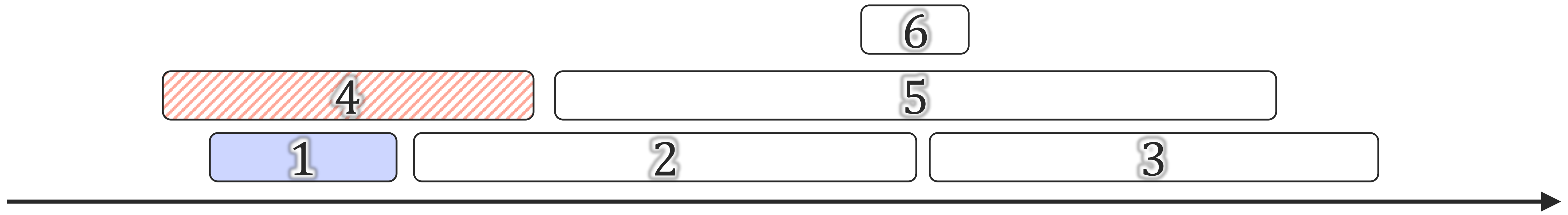

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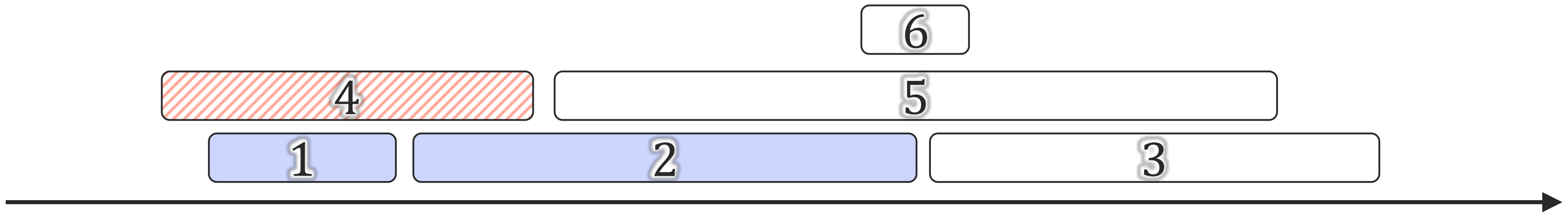

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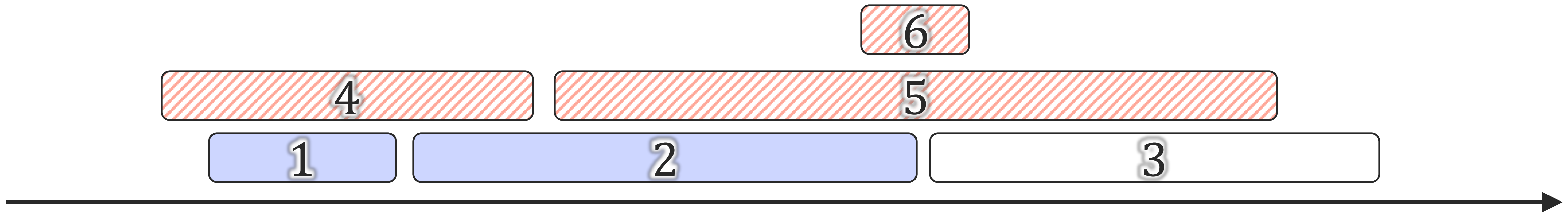

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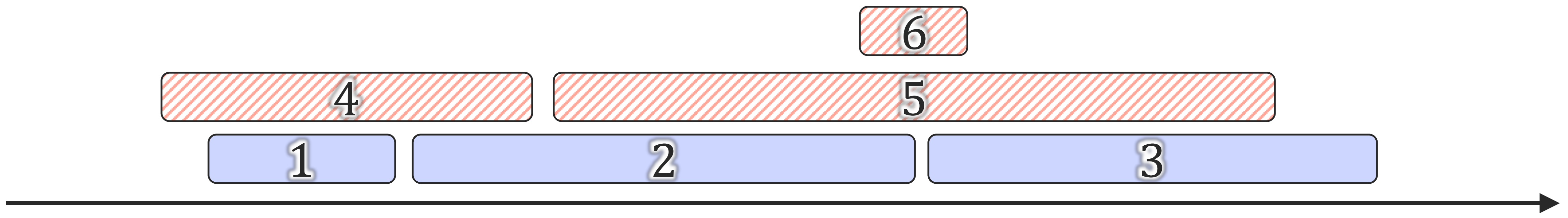

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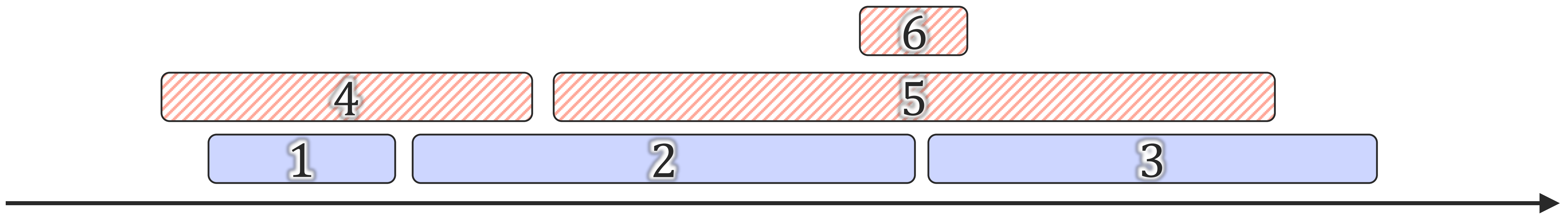


Why is this greedy algorithm correct? We'll see in a minute.

What's the runtime of this algorithm?

- $O(n)$ if the intervals are already sorted by finish time.

→ How? Remember job j that was added last. Next time when looking at candidate job j', only add it if $s_{j'} > e_j$.

- Otherwise $O(n\log(n))$ if we have to sort them by the finish time.

Why does greedy work for interval scheduling?

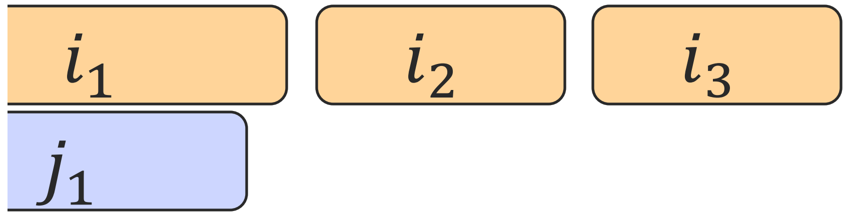
Whenever we make a choice to include an interval in the solution, **we don't rule out an optimal solution.**

→ So, intuitively after we are done, we have an optimal solution.

Intuition for why we never rule out an optimal solution

$$\text{OPT} = \boxed{i_1}, i_2, i_3, \ldots, i_k, i_{k+1}$$

$$\text{Greedy} = \boxed{j_1}, j_2, j_3, \ldots, j_k$$



In OPT: Swap in j_1 for i_1.

Why does greedy work for interval scheduling?

Whenever we make a choice to include an interval in the solution, **we don't rule out an optimal solution.**

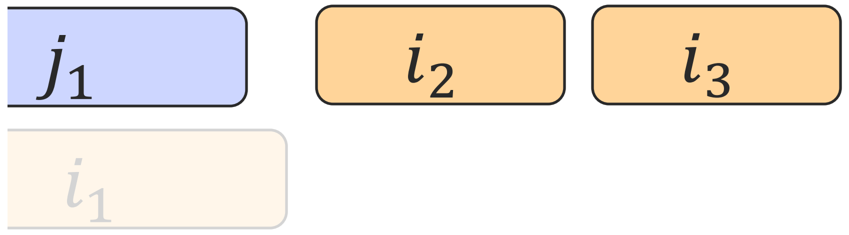
→ So, intuitively after we are done, we have an optimal solution.

Intuition for why we never rule out an optimal solution

$$\text{OPT} = \boxed{\begin{array}{c} i_1 \\ j_1 \end{array}}$$

OPT $= i_1, i_2, i_3, \ldots, i_k, i_{k+1}$

Greedy $= j_1, j_2, j_3, \ldots, j_k$



In OPT: Swap in j_1 for i_1.

More formal argument: Proof by induction

Claim: For any $m \leq k$, **there is an optimal schedule** OPT that agrees with greedy's solution G, on the first m intervals.

Recipe for Greedy Algorithm and Analyses

Greedy makes a series of choices. We show that no choice rules out the optimal solution. How?

Inductive Hypothesis:

→The first m choices of greedy match the first m steps of some optimal solution.

→Or, after greedy makes m choices, achieving optimal solution is still a possibility.

Base case: → At the beginning, achieving optimal is still possible!

Inductive step: **Use problem-specific structure**

If the first m choices match, we can change OPT's $m + 1^{st}$ choice to that of greedy's, and still have a valid solution that no worst than OPT.

Conclusion: The greedy algorithm outputs an optimal solution.

Horn Formula

Variables: $x_1, \dots, x_n \in \{True, False\}$, a literal is x_i or $\overline{x_i}$.

Clauses:

1. "Implication clause" (with no negatived variable)

$$\left(x_i \wedge x_j \wedge \cdots\right) \Rightarrow x_k \longleftrightarrow$$

Equivalent to

$$\overline{x_i} \vee \overline{x_j} \vee \cdots \vee x_k$$

2. "Pure negative clauses"

$$\left(\overline{x_i} \vee \overline{x_j} \vee \cdots\right)$$

Horn Formula: AND of m Horn clauses

Horn Clause’s Significance

Why care about these clauses?

→Used in computational logic, theorem proving, etc.

→Prolog is based on Horn clauses.

Horn-SAT

Implication clause

$(x_i \wedge x_j \wedge \cdots) \Rightarrow x_k$

Pure negative clauses

$(\overline{x_i} \vee \overline{x_j} \vee \cdots)$

Input:

A Horn formula (AND of Horn clauses)

Output:

Find an assignment for the variables that makes the Horn formula True, if such and assignment exists.

Greedy Algorithm for Horn-SAT

Horn-Formula Clauses:

$$(w \wedge y \wedge z) \Rightarrow x$$
$$(x \wedge z) \Rightarrow w$$
$$x \Rightarrow y$$
$$\Rightarrow x$$
$$(x \wedge y) \Rightarrow w$$
$$(\bar{w} \vee \bar{x} \vee \bar{y})$$

Variable assignments:

x

y

z

w

```
For all $i$, set $x_i = False$
While there exists $\left((x_i \wedge \cdots \wedge x_j) \Rightarrow x_k\right) = False$
        Set $x_k = True$
If every pure negative clause $(\overline{x_i} \vee \cdots \vee \overline{x_j}) = True$
        Return $(x_1, \ldots, x_n)$
Else
        Return "not satisfiable"
```

Why does Greedy Work for Horn-SAT?

What's the pattern in this case?

We want to establish that when Greedy sets a variable $x_i = True$, it does not ruin a satisfying assignment.

In fact, we will prove

→ The set of variables set to True by the Greedy algorithm, are also set to True in any satisfying assignment.

Proof of Claim

Claim: The variables set to True by Greedy, are also True in the satisfying solution.

Proof: By induction on the iteration of the While loop

Base case: In the 0th iteration of the While loop, nothing is set to True.

Induction hypothesis: The first m variables set to True by Greedy are also True in every satisfying solution.

Inductive step:

- Let x_{m+1} be the $m + 1^{st}$ variable set to True by Greedy.

→ Only happens if there was an unsatisfied implication $(x_i \wedge \cdots \wedge x_j) \Rightarrow x_{m+1}$ before the $m + 1$ iteration of the while loop. i.e., $x_{m+1} = False$ and $(x_i \wedge \cdots \wedge x_j) = True$.

→ If $(x_i \wedge \cdots \wedge x_j) = True$ before the m+1 iteration of Greedy, then $(x_i \wedge \cdots \wedge x_j) = True$ also in the satisfying solution.

- The only way to satisfy this clause in SAT is to also have $x_{m+1} = True$.

Horn-SAT Proof completed

Claim: The greedy solution is correct.

1) If Greedy outputs a solution, then the solution is satisfiable.

This is true because the While loop and If condition check that all clauses are satisfied

2) If the Horn Formula is satisfiable, then Greedy outputs a satisfiable solution.

Assume to the contrary that this is not true. So, Greedy outputs "unsatisfiable" even though a satisfying assignment exists.

→In Greedy's assignment, there is a violated pure clause $(\bar{x}_i \vee \cdots \vee \bar{x}_j)$

→So, every variable in this clause is set to True

→By previous slide, these variables are also set to True in any satisfiable solution and this clause is also violated by the satisfying assignment

→Contradiction!

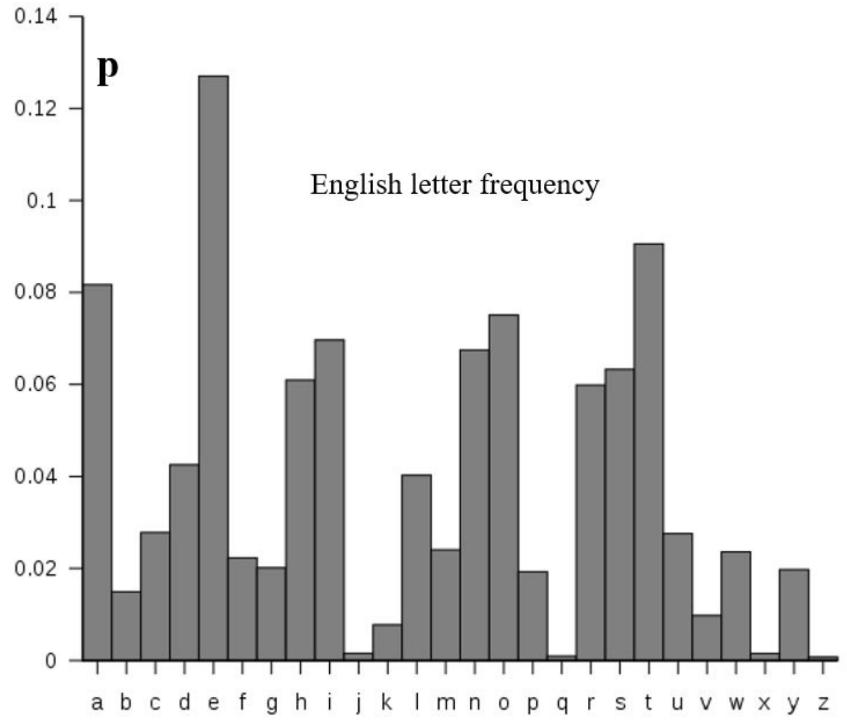
Next up: Codes!

Data Compression and Encoding

Common encodings of English characters use a fixed length of code per character.

If the goal is to save space, can we encode the alphabet better?

- If we know which letters are more common
- Use shorter codes for very common characters (like e, a, s, t).



Example of encodings

Assume we just have 4 letters, A, B, C, D with associated frequencies.

Freq.	Letter	Encoding #1	Encoding #2	Encoding #3
0.4	A			
0.2	B			
0.3	C			
0.1	D			
Total cost				

Example of encodings

Assume we just have 4 letters, A, B, C, D with associated frequencies.

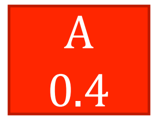
Freq.	Letter	Encoding #1	Encoding #2	Encoding #3
0.4	A	00	0	0
0.2	B	01	00	110
0.3	C	10	1	10
0.1	D	11	01	111
Total cost		$2N$	$(0.4 + 0.3) \times N + (0.1 + 0.2) \times 2N = 1.3N$	$0.4 \times N + 0.3 \times 2N + (0.2 + 0.1) \times 3N = 1.9N$

But encoding #2 is lossy: What does 000 represent? AB or BA?

Encoding #3: No code is a prefix of another.

→ There is only one way to interpret any code.

Any Prefix codes and Trees

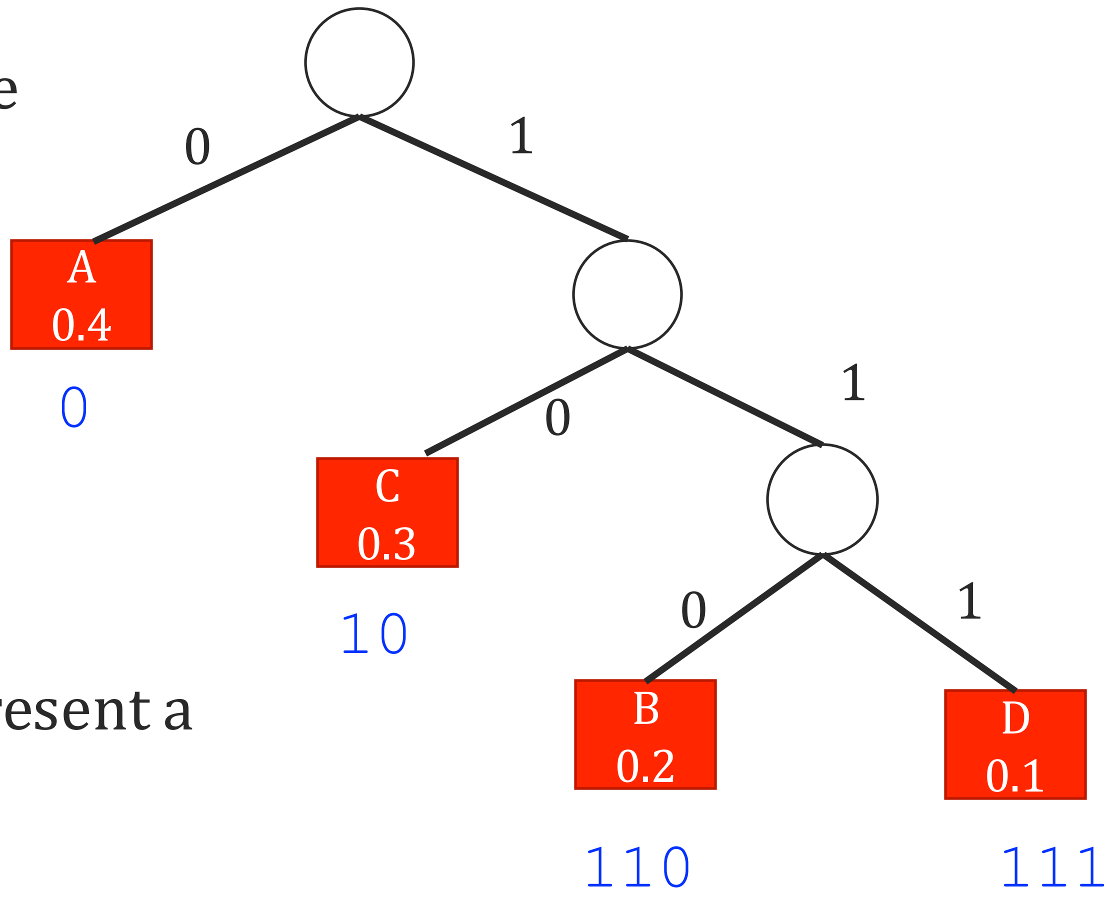


means "A" has freq. 0.4.

Prefix free code: No code x is a prefix of another code z.

Any prefix-free code on n letters can be represented as a binary tree with n leaves.

- Leaves indicate the coded letter
- The code is the "address" of a letter in the tree



Any tree with the letters at the leaves, also represent a prefix-free code.

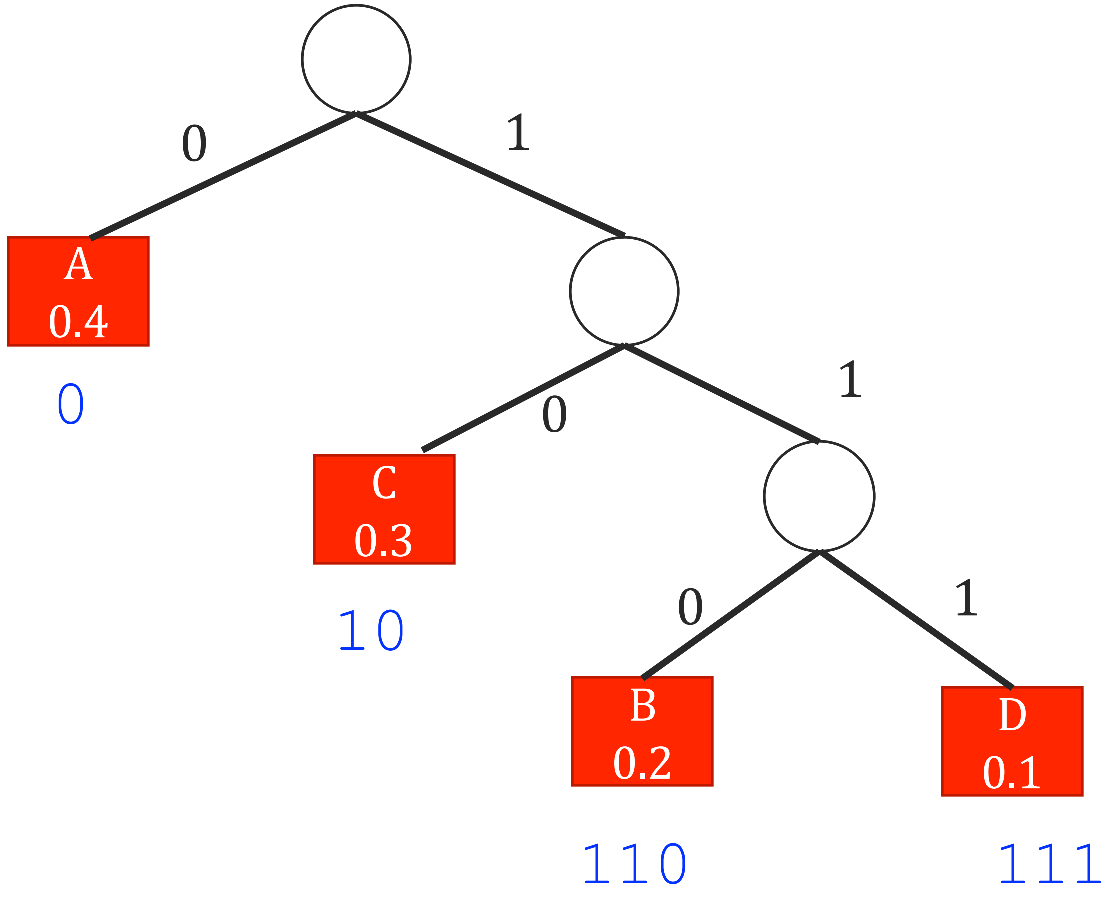
Tree and Code Size

means “A” has freq. 0.4.

Imagine we are encoding a length N text:
→ that is written in n letters with frequencies $f_1, f_2, \dots, f_n$.

How long is the encoded message?

$$\text{length of encoding} = \sum_{i=1}^{n} N \cdot f_i \cdot \text{len}(\textit{encoding } i)$$



Definition: Cost of a prefix-code/tree is

$$\text{Cost(tree)} = \sum_{i=1}^{n} f_i \cdot \text{depth}(\textit{leaf } i)$$

Optimal Prefix-free Codes

Input: n symbols with frequencies $f_1, \dots, f_n$

Output: A tree (prefix-free code) encoding.

Goal: We want to output the tree/code with the smallest cost

$$\text{Cost(tree)} = \sum_{i=1}^{n} f_i \cdot \text{depth}(leaf\ i)$$

Wrap up

Greedy Algorithms are simple to design

A bit harder to analyze perhaps!
→ Induction is our friend.
→ Find a nice substructure of the problem

Next time

- More on Huffman
- Minimum Spanning Trees