LECTURE 20

* Reductions recap
* NP-completeness
* Independent Set \( \leq_p \) Integer Programming
* 3SAT \( \leq_p \) Independent Set
Problem A \leq_p Problem B

1) Reduction Algorithm (converts inputs to A \rightarrow inputs to B)

2) \exists A solution to original input to A \Rightarrow \exists a solution to input to B

\exists a solution to _B \Rightarrow \exists a solution to A

Problem A is no harder than Problem B.
Remarks:

1) Reduction algorithm needs to run in polytime

\[ A \leq^* B \]

2) \[ A \leq_p B \land B \leq_p C \implies A \leq_p C \]

3) \[ A \& B \leq_p C \] - both are problems with no knowledge
NP-Complete Problems

A problem $A$ is NP-complete if

1) $A \in \text{NP}$

2) Every problem $B \in \text{NP}$ reduces to $A$

$B \leq_p A$

Corollary: If $A$ & $B$ are NP-complete

$A \leq_p B \ & \ B \leq_p A$

Corollary: If exists a polytime alg for some NP-complete problem

$\Rightarrow$ $\text{NP} = \text{P}$
To show: Problem $A$ is NP-complete

1) $A \in \text{NP}$ [Exhibit a verification algorithm]

2) Pick some well known NP complete problem

say $3\text{SAT} \leq_p A$

Show that $3\text{SAT} \leq_p A$
NP complete problems: "every NP problem in" ALL OF NP

[Cook70]

CIRCUIT SAT

3SAT

INDEPENDENT SET

INTEGRAL PROGRAM

CLIQUE

VERTEX COVER

RUORATA CYCLE (directed)
Definition: (Independent Set)

Given a graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is an independent set if there are no edges inside $S$, i.e., $\forall u, v \in S$, $(u, v) \notin E$. 
**INTRODUCTION:**

**Input:** Graph $G=(V,E)$, $K$

**Solution:** An independent set of size $K$

**PROBLEM:**

$G=(V,E)$, $K$

**CONVERSION TO INTEGER PROGRAMMING**

**Input:** A linear program

**Solution:** An integer solution to linear program

$x_i = \begin{cases} 
1 & \text{if } i \in \text{Independent Set} \\
0 & \text{otherwise}
\end{cases}$

$\mathbf{IP}$

$0 \leq x_i \leq 1$

$\sum_{i=1}^{K} x_i = K$

$\forall (i,j) \in E \quad x_i + x_j \leq 1$

1) Run in polytime

2) To PROVE:

a) $G$ has ind set of size $K$

$\implies$ exists a solution to IP

b) Exists a solution to IP $\implies$ $G$ has an indset of size $K$
3SAT: fundamental NP-complete

INPUT: 1) Boolean variables $x_1, \ldots, x_n \in \{0, 1\}$

2) Clauses: $\rightarrow (x_1 \lor \overline{x}_2 \lor x_7) \land \rightarrow (x_5 \lor \overline{x}_6 \lor \overline{x}_8) \land \rightarrow (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_3) \land \}

SOLUTION: An assignment
\{x_1, \ldots, x_n\} \rightarrow \{0, 1\}
that satisfies all the clauses
**3SAT**

**INPUT:** 3SAT formula on $x_1, x_n$

$$(x_1 V x_2 V x_3) \land (\overline{x_1} V x_2 V x_3) \land \ldots$$

**SOL:** A satisfying assignment

$$x = 1, y = 1, z = 0, w = 0$$

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**IND SET**

**INPUT:** Graph $G = (V, E)$

**SOL:** An independent set of size $K$

$K = \# of clauses
1) A clause $\bar{x} \lor \bar{y} \lor \bar{z}$

\[ \Downarrow \]

create a $\Delta 1 e$

In the satisfying assignment if $\bar{x} \lor \bar{y} \lor \bar{z}$

has true literals $\bar{x}, \bar{y} \Rightarrow$ add one of those to independent

2) If variable $x$,

add an edge between every vertex labeled $x$

to every vertex labeled $\bar{x}$
Proof:

1) \exists a satisfying assignment

\[ \Rightarrow \exists a \text{ independent set of size } K \]

Proof: If \( \{x_1 \ldots x_n \} \Rightarrow \{0,1\} \) satisfies the formula then \( \forall \) each clause \( x_i \lor \overline{x_j} \lor \overline{x_k} \), pick some true literal, include the vertex in independent set

\[ \Rightarrow |\text{Independent Set}| = \# \text{ of classes} \]

\[ \Rightarrow \text{No edges inside.} \]
\( \exists \) an independent set in \( G^- \) with size \( K = \# \) triangles

\[ \forall \text{ variable } x_i = \begin{cases} 1 & \text{if some vertex } x \in \text{ Ind Set} \\ 0 & \text{if some vertex } x \in \text{ Ind Set} \end{cases} \]

arbitrary otherwise

Ind Set has \( \overline{x_i} \) then \( \overline{x_i} \notin \text{ Ind Set} \)

\( \forall \) Ind set picks exactly one vertex in each \( \triangle \)

\( \Rightarrow \) every clause has 1 satisfying literal