Zero Sum Games

Ex: Rock-Paper-Scissors (R-P-S)

2 Players (call them Row and Col)

Both pick one of R, P, or S; who wins?

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Each entry of “utility matrix” says how much Row wins, = \( R_{ij} \) = how much Col loses (reason for name “Zero Sum”)

Row wants to maximize value of entry
Col wants to minimize value of entry

What is best strategy for Row? for Col?
What does “play the game” mean? (1/3)

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Col\[Row\] wants to \[maximize\] entry \[minimize\] entry

1) Row picks R, P or S, tells Col, then Col picks 
   Col always wins

2) Col picks R, P or S, tells Row, then Row picks 
   Row always wins

3) Row and Col pick, then announce at same time 
   What is best strategy for Row, or Col?
What does “play the game” mean? (2/3)

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Col \( \rightarrow \) Player wants to \[ \begin{bmatrix} \text{maximize} \\ \text{minimize} \end{bmatrix} \) entry

3) Row and Col pick, then announce at same time
3a) Row (almost) always picks same row
Col notices this, can (almost) always win
3b) Col (almost) always picks same column
Row notices this, can (almost) always win
What does “play the game” mean? (3/3)

Row: \( i = \begin{array} { c } { 1 } & { R } & { 0 } \backslash { 1 } \\ { 2 } & { P } & { 1 } \backslash { 0 } \\ { 3 } & { S } & { -1 } \backslash { 1 } \end{array} \) \( x_i \) \( [\text{Row}] \) \[ \text{wants to} \] \[ \text{minimize} \] entry

Col

\[ j = \begin{array} { c } { 1 } & { R } & { X } \\ { 2 } & { P } & { O } \\ { 3 } & { S } & { X } \end{array} \] \( x_j \) \[ \text{Announce at same time} \]

3) Row and Col pick, then announce at same time

Row picks row \( i \) with probability \( x_i \)

Col picks column \( j \) with probability \( y_j \)

Let \( U = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \) = “utility”

Probability of picking \( U(i,j) = x_i \cdot y_j \)

Expected utility = \( EU = \sum U(i,j) \cdot x_i \cdot y_j \)

How should \([\text{Row}]\) pick \([\text{xi}]\) to \([\text{maximize}]\) \( EU \)
Choosing a “strategy” for game

- Row's strategy = \( x = (x_1, x_2, x_3) \)
- Col's strategy = \( y = (y_1, y_2, y_3) \)

\[
\mathbb{E}U = \sum_{i,j} U(i, j) \cdot x_i \cdot y_j = \frac{1}{3} \sum_{i,j} U(i, j) = \frac{1}{3} \sum_{i,j} y_j \cdot 0 = 0
\]

- Ex: \( x = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \ y = (0, 0, 1) \)

- Similarly, if \( x = (x_1, x_2, x_3), \ y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) then \( \mathbb{E}U = 0 \)

- Both Row and Col have strategies that guarantee the other player wins 0 in expectation

- Value of game = \( \mathbb{E}U = 0 \), with solution \( (x, y) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)
Let's try a different game

<table>
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<td>5</td>
</tr>
<tr>
<td></td>
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<tr>
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- Is \(x = (\frac{1}{2}, \frac{1}{2})\) still a good strategy for Row?
  - if Col picks \(L\) then \(\text{EU} = \left[\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot (-1) = 2\right]\)
  
  \[\Rightarrow \text{Col should choose } R, \text{ loses } \text{EU} = -1, \text{ wins +1}\]

- Is \(y = (\frac{1}{2}, \frac{1}{2})\) still a good strategy for Col?
  - if Row picks \(T\) then \(\text{EU} = \left[\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot (-3) = 1\right]\)

  \[\Rightarrow \text{Row should choose } T, \text{ wins } \text{EU} = 1\]

- Not like RPS, expect a better strategy
Let's try a different game

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Are there better strategies for Row and Col than \( \frac{1}{2}, \frac{1}{2} \)?

- **Suppose Row plays** \( x = \left( \frac{1}{5}, \frac{4}{5} \right) \)
  
  if Col picks \( \begin{bmatrix} L \\ R \end{bmatrix} \) then \( \text{EU} = \left[ \frac{1}{5} \cdot 5 + \frac{4}{5} \cdot (-1) = \frac{1}{5} \right] \)

  Row wins \( \frac{1}{5} \), no matter what Col's strategy is.

- **Suppose Col plays** \( y = \left( \frac{2}{5}, \frac{3}{5} \right) \)
  
  if Row picks \( \begin{bmatrix} T \\ B \end{bmatrix} \) then \( \text{EU} = \left[ \frac{2}{5} \cdot 5 + \frac{3}{5} \cdot (-1) = \frac{1}{5} \right] \)

  Col loses \( \frac{1}{5} \) no matter what Row's strategy is.

- **Value of game** = \( \text{EU} = \frac{1}{5} \) with solution \( (x, y) \)
Solving a Zero Sum Game as a LP (1/2)

Rephrase finding the best strategies for Row and Col as dual LPs

- **Row's goal**: choose \( x = (x_1, x_2) \) to maximize payoff from Col's best response:
  - pick \( x \) to maximize \( \min(L - 5x_1 - x_2, -3x_1 + x_2) \)

- **Convert to LP**: for Row
  - constraints: \( x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1 \)
  - compute \( \min(x) \):
    - maximize \( z \): \( z \leq 5x_1 - x_2, z \leq -3x_1 + x_2 \)
  - LP:
    - maximize \( z = 1 \cdot z + 0 \cdot x_1 + 0 \cdot x_2 \)
    - s.t. \( x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1, z \leq 5x_1 - x_2, z \leq -3x_1 + x_2 \)
Solving a Zero Sum Game as a LP (2/2)

Replicate finding the best strategies for Row and Col as dual LPs

- Col's goal: choose \( y = (y_1, y_2) \) to minimize payoff from Row's best response:
  - Pick \( y \) to minimize \( \max(5y_1 - 3y_2, -x_1 + x_2) \)

- Convert to LP:
  - Constraints: \( y_1 \geq 0, y_2 \geq 0, y_1 + y_2 = 1 \)
  - \( \text{max}(\cdot) \) minimize \( w: w \geq 5y_1 - 3y_2, w \geq -y_1 + y_2 \)
  - LP: minimize \( w = 1 \cdot w + 0 \cdot y_1 + 0 \cdot y_2 \)
    - s.t. \( y_1 \geq 0, y_2 \geq 0, y_1 + y_2 = 1, w \geq 5y_1 - 3y_2, w \geq -y_1 + y_2 \)
These Two LPs are Dual

- Row's LP: maximize
  \[ Z = 1 \cdot z + 0 \cdot x_1 + 0 \cdot x_2 \text{ s.t.} \]
  \[ x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 = 1, \]
  \[ z \leq 5x_1 - x_2, \ z \leq -3x_1 + x_2 \]

- Write \( z = z_1 - z_2, \ z_1 \geq 0, \ z_2 \geq 0 \)
  \[ \text{max} \ z = \begin{bmatrix} 1, & -1, & 0, & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{bmatrix} = c^T x \]
  s.t.
  \[ z_1 - z_2 - 5x_1 + x_2 \leq 0 \]
  \[ z_1 - z_2 + 3x_1 - x_2 \leq 0 \]
  \[ x_1 + x_2 \leq 1 \]
  \[ -x_1 - x_2 \leq -1 \]

- Column's LP: minimize
  \[ w = 1 \cdot w + 0 \cdot y_1 + 0 \cdot y_2 \text{ s.t.} \]
  \[ y_1 \geq 0, \ y_2 \geq 0, \ y_1 + y_2 = 1 \]
  \[ w \geq 5y_1 - 3y_2, \ w \geq -y_1 + y_2 \]

- Write \( w = w_1 - w_2, \ w_1 \geq 0, \ w_2 \geq 0 \)
  \[ \text{min} \ w = \begin{bmatrix} 0, & 0, & 1, & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{bmatrix} = b^T y \]
  s.t.
  \[ b^T \begin{bmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

\[ \text{max} \ c^T x \text{ s.t.} \ A \cdot x \leq b \]
\[ \text{min} \ b^T y \text{ s.t.} \ A^T y \geq c \]

Dual \( \Rightarrow \) Row's and Column's strategies match: \( z = c^T x = b^T y = w \)
General Approach to Zero Sum Games

- Game specified by \( m \times n \) matrix \( U \)
- Row strategy: \( x = (x_1, x_2, \ldots, x_m)^T, \ x_i \geq 0, \ \sum x_i = 1 \)
- Col strategy: \( y = (y_1, y_2, \ldots, y_n)^T, \ y_j \geq 0, \ \sum y_j = 1 \)

- Expected Utility: \( EU = \sum_{i,j} U(i,j) \cdot x_i \cdot y_j = x^T U y \)

Row's Goal: pick \( x \) to maximize \( \min(x^T U) \)
Col's Goal: pick \( y \) to minimize \( \max(Uy) \)

\[
\text{Maximize } z: \quad z \leq (x^T U); \quad j = 1:n \\
x_i \geq 0; \quad i = 1:m \quad \sum x_i = 1 \\
\text{Minimize } w: \quad w \geq (Uy); \quad i = 1:m \\
y_j \geq 0; \quad j = 1:n \quad \sum y_j = 1
\]

Dual: \( \max z = \min w = \text{Value of Game} \)
Solve using simplex...
Game Theory - A little history

• Von Neumann - before duality of LPs

• Many variations:
  more than 2 players
  nonzero/zero sum
  perfect/imperfect information (poker)
  combinatorial/non combinatorial (chess)
  discrete vs continuous (robot motion planning)

• Many Applications
  economics, computational complexity, sociology, biology, philosophy...

• Many Prizes
  5 Nobel Prizes (1 recipients) Economics
  4 Oscars - A Beautiful Mind - John Nash