Lower Bound on Communication for Matrix Multiplication

Outline

- We have a CPU, cache of size M, and large memory
- We want to multiply n x n matrices C = A*B, which are too big to fit in cache
- Moving data between memory and cache is expensive
- Is there a lower bound on how many reads and writes (moving words between cache and memory) are needed to perform C = A*B?
- Is there an optimal algorithm that attains this lower bound?

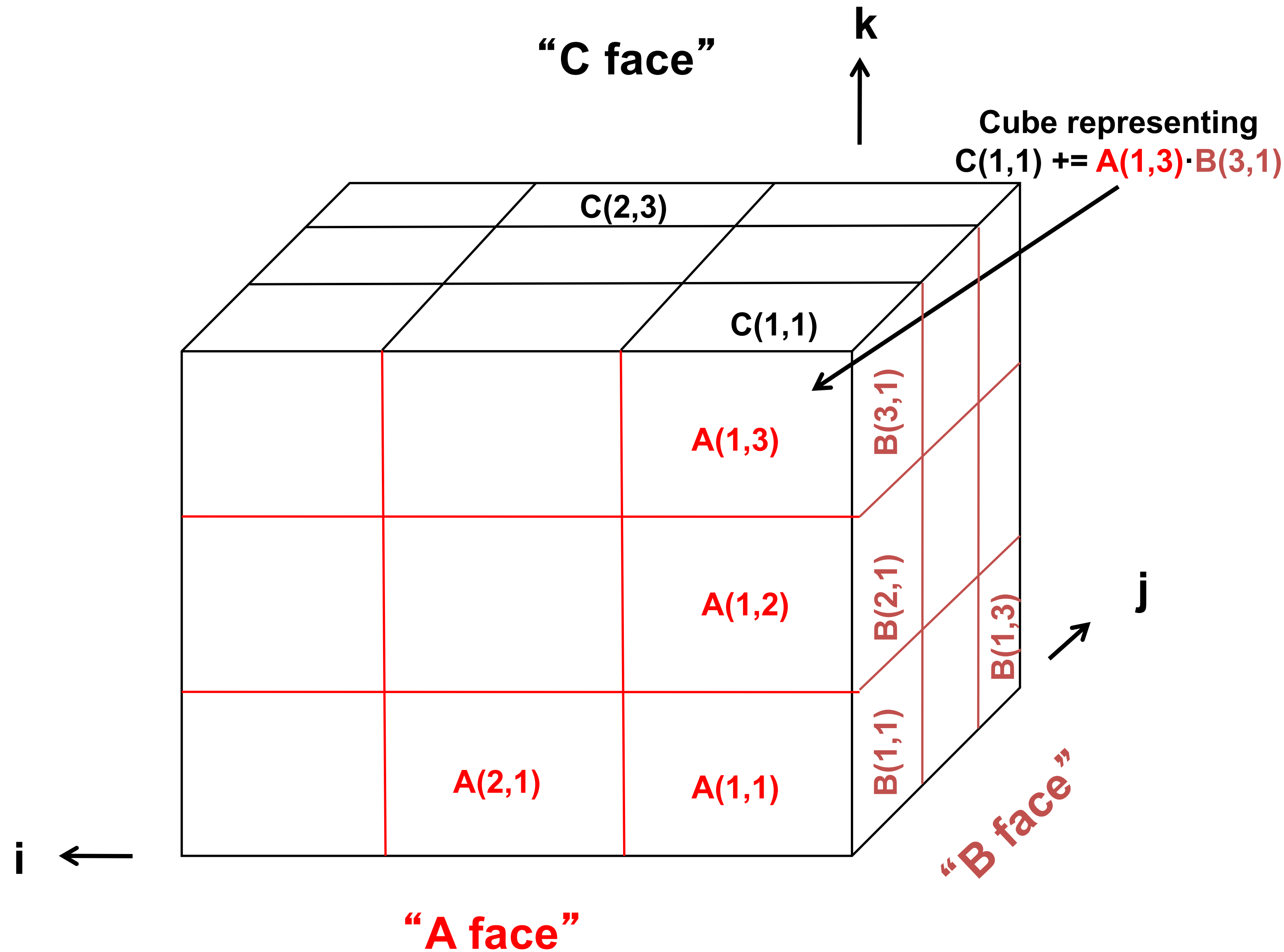
Results

- Thm (Hong, Kung, 81) A lower bound on the number of reads and writes is $\Omega(n^3/\sqrt{M})$
- This lower bound is attainable by ”tiling” the 3 nested loops (working on square submatrices of A, B and C that all fit in cache simultaneously)
- Both results can be extended to
  - more general code that looks like nested loops accessing arrays (more linear algebra, tensors, CNNs, …)
  - Memory hierarchies
  - Moving data between processor on a network
- See bebop.cs.berkeley.edu for more details

Lower Bound Proof Sketch

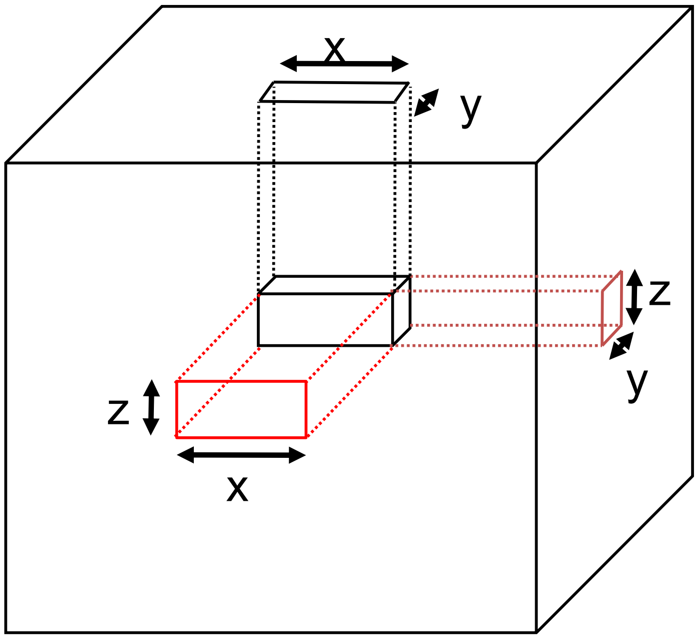
- Inner loop of matmul: C(i,j) += A(i,k)*B(k,j)
- Performing one inner loop iteration requires 3 words be in cache
- If I can only fit M words in cache, how many iterations can I do?
- Hard part (next slide): find an upper bound F on the number of iterations I can do
- Need to do n^3 iterations => need to refill cache n^3/F times => #words moved $\geq \left(\frac{n^3}{F}\right) M$

Model iterations over (i,j,k) as an n x n x n cube



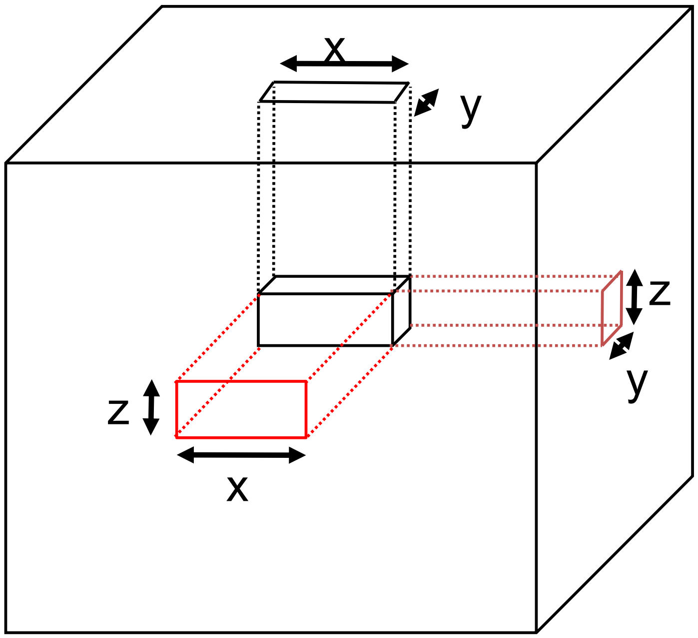
- If we have at most M "A squares", "B squares", and "C squares" on faces, how many cubes can we have?

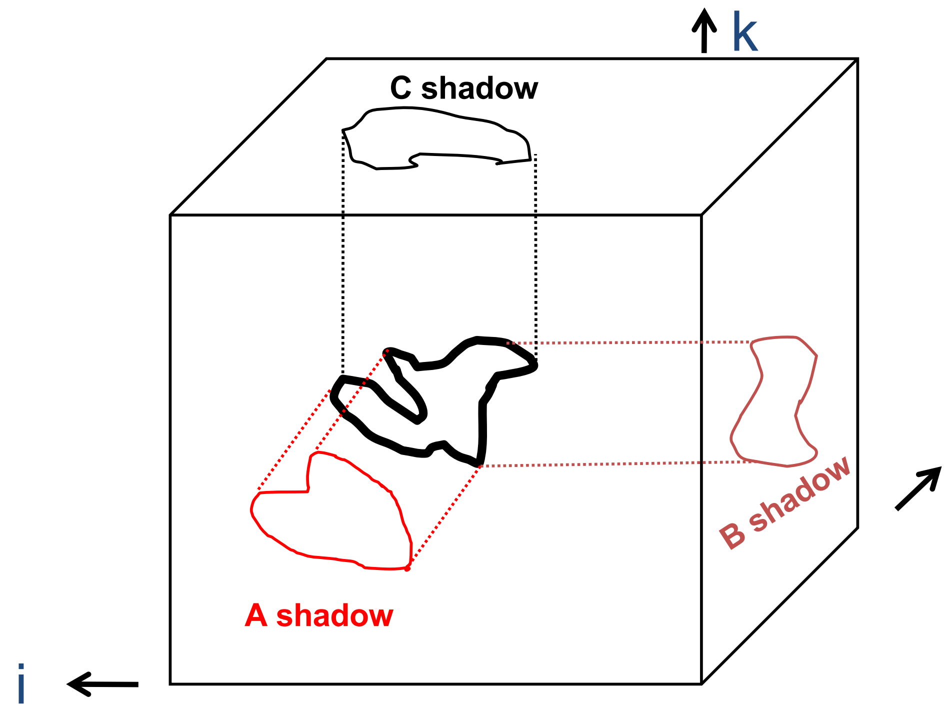
If I only have M squares, how many cubes can I "cover"?



 # cubes in black box with side lengths x, y and z
= Volume of black box
= x·y·z
= (xz · zy · yx)$^{1/2}$
= (#A☐s · #B☐s · #C☐s)$^{1/2}$

If I only have M squares, how many cubes can I “cover”?





(i,k) is in A shadow if (i,j,k) in 3D set
(j,k) is in B shadow if (i,j,k) in 3D set
(i,j) is in C shadow if (i,j,k) in 3D set

 # cubes in black box with side lengths x, y and z
= Volume of black box
= x·y·z
= $(xz \cdot zy \cdot yx)^{1/2}$
= $(\#A\square s \cdot \#B\square s \cdot \#C\square s)^{1/2}$

Thm (Loomis & Whitney, 1949)
 # cubes in 3D set = Volume of 3D set
≤ $(\text{area}(A\ \text{shadow}) \cdot \text{area}(B\ \text{shadow}) \cdot \text{area}(C\ \text{shadow}))^{1/2}$

Finishing the lower bound proof

- F = bound on # of loop iterations with M words

  = bound on #cubes with shadows of size M

  $\leq$ (#entries_A*#entries_B*#entries_C)$^{1/2}$

  $\leq$ $(M \ * M \ * M)^{1/2}$ = $M^{3/2}$
- #words moved $\geq (n^3/F)M = \ n^3$ / $\sqrt{M}$