Bipartite Matching Problem (BM)

**Input:** Bipartite Graph \( G = (L, R, E) \) with \( E \subseteq L \times R \)

**Defn:** A matching is a set of edges \( M \subseteq E \) s.t.
- no pair of edges in \( M \) touches the same vertex.

**Goal:** find maximum matching, i.e., \( \max |M| \)
- s.t. \( M \subseteq E \) is a matching.

We show how to solve BM using an algorithm for MaxFlow.
What if a flow gets split?

Fact: If all capacities are integers then there a max flow.

$|M|$ matching $M$ on a bipartite graph $G$

$\text{val}(f)$ integral flow on network $G$
Claim: Suppose $M$ is a matching in $G$. Then $f$ integral flow on $G$ with $\text{val}(f) = |M|$. 

Proof:

Push 1 unit of flow on edges in $M$.

$L(M) =$ vertices in $L$ touching $M$

$R(M) =$ " R " M.

Push 1 unit of flow from $s$ to $v$, $\forall v \in L(M)$.

Push 1 unit of flow from $u$ to $t$, $\forall u \in R(M)$.

$\text{val}(f) = |L(M)| = |M|$.

$\square$
Claim: $f$ is an integral flow on $\tilde{G}$

Then, $\exists$ a matching $M$ on $G$ s.t. $|M| = \text{val}(f)$.

Proof: Since capacities in $\tilde{G}$ are all 1, the flow on each edge could be either 0/1.

$$M = \{ (u,v) : u \in L, v \in R, f_{u,v} = 1 \}.$$
The notion of a reduction

"a problem \( A \) reduces to a problem \( B \) if any subroutine to solve \( B \) can be used to solve \( A \)"

In more detail:

\[ \begin{align*}
\text{pre-processing} & \quad \text{efficient} \\
\text{efficient} \quad & \quad \text{efficient}
\end{align*} \]

\( x \rightarrow \quad \text{y} \rightarrow \quad \text{B} \rightarrow \quad B(y) \rightarrow \quad \text{A}(x) \rightarrow \)

\( \circ \) an efficient alg for \( B \) \( \Rightarrow \) an efficient alg for \( A \).

A reduction = pre-processing + post-processing.

\( \circ \) If an efficient alg for \( B \) \( \Leftrightarrow \) an efficient alg for \( A \)
Matrix Multiplication Strassen

\[
\log_2 n = n^{\log_2 7} \approx n^{2.8}
\]

\[
\begin{array}{c|c|c}
\hline
n & \sqrt{n} & \sqrt{n} \\
\hline
A & n & B \\
\hline
\end{array}
\]

\[
C = n \\
\sqrt{n} \times \sqrt{n}
\]

want to compute

\[
A^{-1} \\
\sqrt{n}
\]

\[
A \cdot A^{-1} = I
\]

Matrix Mult \rightarrow Matrix Inverse

\[
\begin{bmatrix}
A & B \\
\end{bmatrix}_{n \times n}
\]

\[
\begin{bmatrix}
I_n & A & 0 \\
0 & I_n & B \\
0 & 0 & I_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
I - A & AB \\
0 & I - B
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & I
\end{bmatrix}
\]
Rudrata Cycle

$G = (V, E)$ undirected
find a cycle that visits all vertices exactly once.

Rudrata path

$G = (V, E)$ undirected, source $s$, target $t$
goal: find a path from $s$ to $t$ that visit all vertices exactly once.
If $G$ has a Rudrata path from $s$ to $t$,

$\Rightarrow G'$ has a Rudrata cycle.

$\Rightarrow G$ has a Rudrata path from $s$ to $t$.  

If $G$, $s$, $t$ Rudrata Path

Rudrata Cycle

$G'$

$G$

$x$ $s$ $t$

$\Rightarrow$ Rudrata Path

Rudrata Cycle

$G$

$x$ $s$ $t$

$G'$

$\Rightarrow$ Rudrata Cycle

$G'$

$G$

$x$ $s$ $t$

$\Rightarrow$ Rudrata Path

Rudrata Cycle