Lecture 21

CS 170

- Reductions

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Search problems

A **search problem** is specified by an algorithm VERIFY that takes two inputs, an instance I and a proposed solution S and runs in time polynomial in $|I|$.

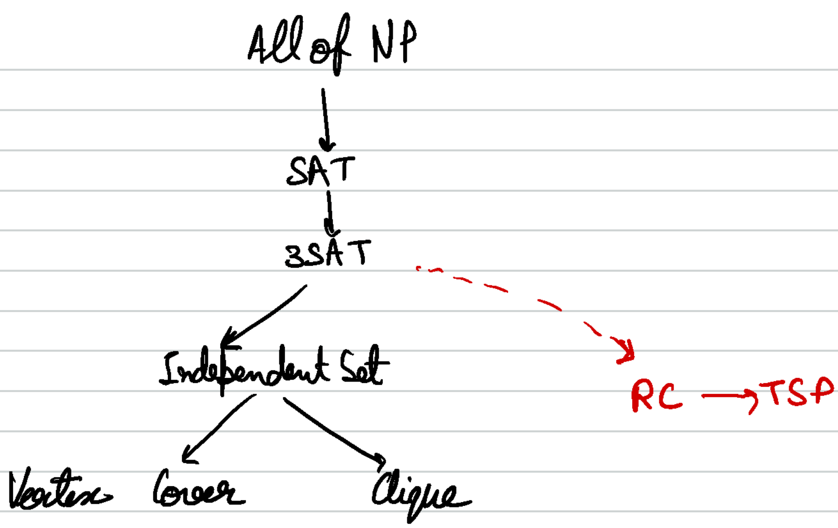
We say that S is a solution to I if VERIFY$(I, S) = 1$

P : Class of search problems where we can **find** a solution in polynomial in |I| time.

NP : Class of all search problems; i.e. the class of all problems where we can **verify** a solution in polynomial time

NP- completeness

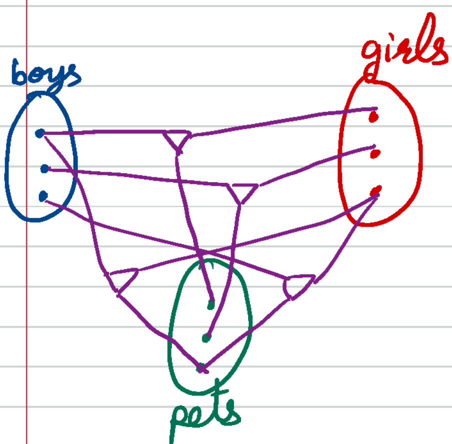
A search problem is NP-complete if all other search problems reduce to it.

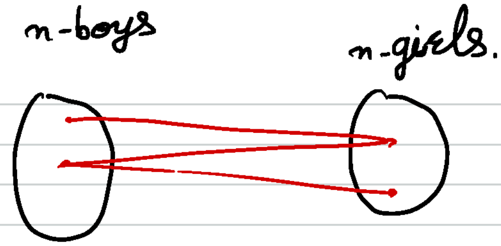


3D Matching

Input: n boys, girls and pets with preference triples $\{(b, g, p)\}$

Solution: n-disjoint triples.





Input: n boys & n girls preference tuples $\{(b, g)\}$

Solution: n-disjoint tuples.

3SAT $\longrightarrow$ 3DM

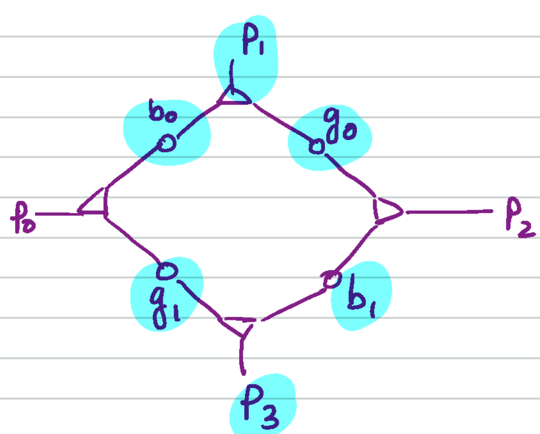
Instance: Formula ϕ on variable $x_1 \dots x_m$

Solution: A satisfying assignment

Input: n boys, girls and pets with preference triples $\{(b, g, p)\}$

Solution: n-disjoint triples.

Gadgets



(b_0, g_0, p_1) (b_1, g_1, p_3)

false $\longrightarrow$ p_0 & p_2 free

true $\longrightarrow$ p_1 & p_3 free

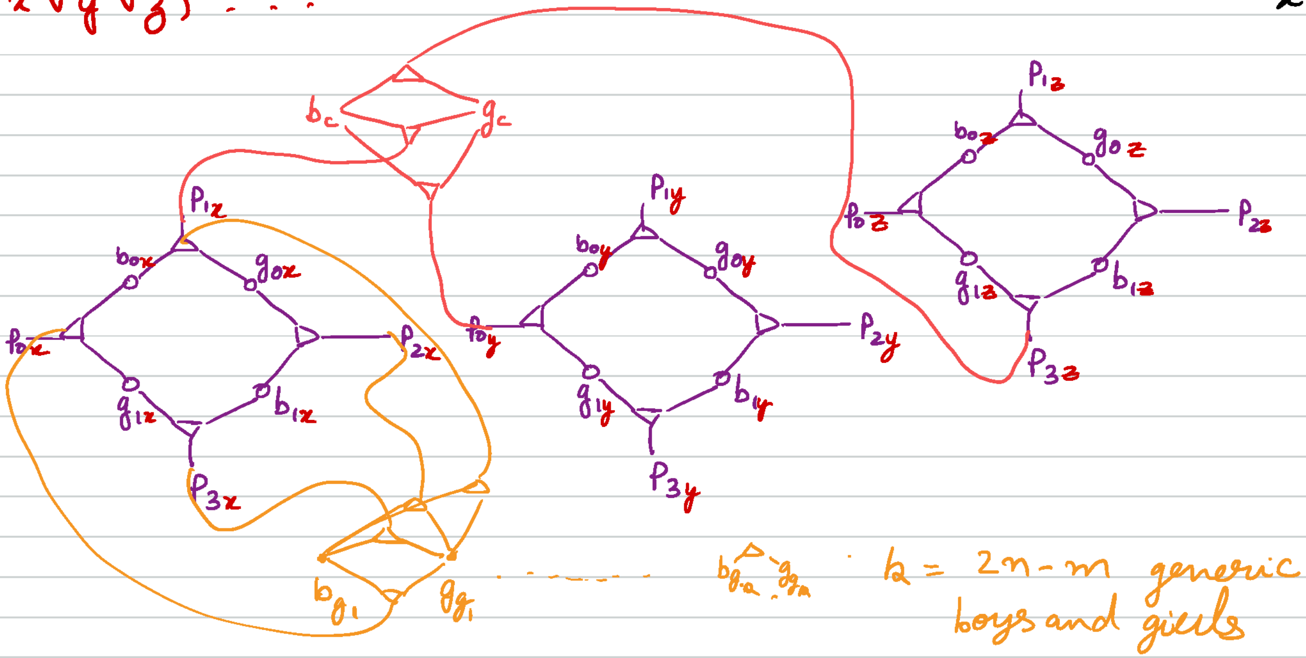
(b_0, g_1, p_0) & (b_1, g_0, p_2)

w.log. $\phi = (x_1 \ldots)(\quad x_2 \quad)(\cdots \bar{x}_3 \quad) \ldots (x_k \quad)$

$(\bar{x}_1 \vee x_2)(\bar{x}_2 \vee x_3) \ldots \quad (\bar{x}_k \vee x_1)$ $\quad x \to x, x$

$\bar{x}, \bar{x}$

$c = (x \vee \bar{y} \vee z) \ldots$

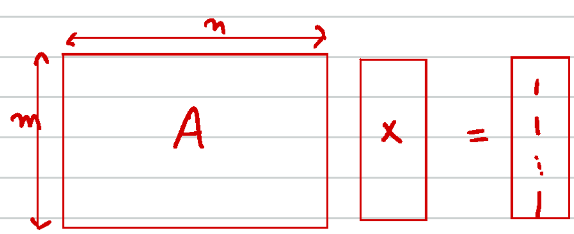


$k = 2n - m$ generic boys and girls

Zero-One Equations (ZOE)

Instance: $A \in \{0,1\}^{m \times n}$

Solution: $x \in \{0,1\}^n$ s.t. $Ax = \mathbb{1}$



3DM $\longrightarrow$ ZOE

Input: t-boys, girls and pets with preference triples $\{(b,g,p)\}$

Solution: t-disjoint triples.

Instance: $A \in \{0,1\}^{m \times n}$

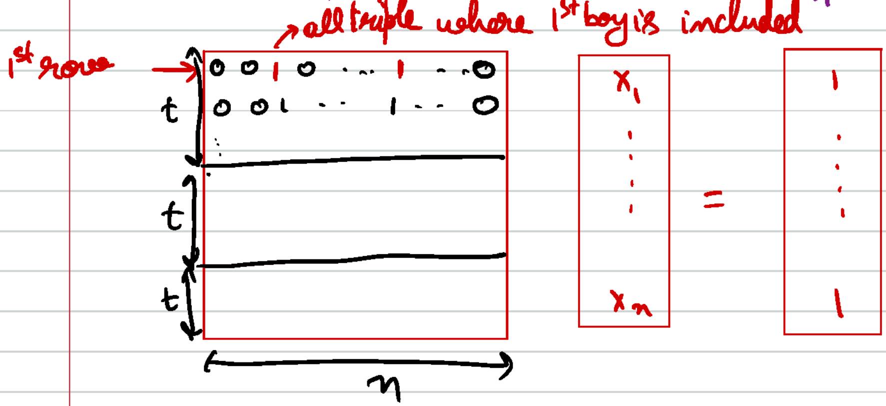
Solution: $x \in \{0,1\}^n$ s.t. $Ax = \mathbb{1}$.

$x_i = 0$ if T_i is not part of the 3DM solution

& $x_i = 1$ if T_i is a part of the solution

$T_1 \quad T_2 \quad \cdots \quad T_n$

$x_1 \quad x_2 \quad \cdots \quad x_n$



ZOE $\longrightarrow$ RC / HC

(1) ZOE $\longrightarrow$ RC with paired edges (RCwPE)

(2) RCwPE $\rightarrow$ RC

RCwPE

Instance: $G = (V, E)$ $\quad C \subseteq E \times E$

Solution: RC S in G s.t. $\forall (u,v) \in C$ exactly one of u, v is in S.

ZOE $\longrightarrow$ RC w PE

ZOE

Instance: $A \in \{0,1\}^{m \times n}$

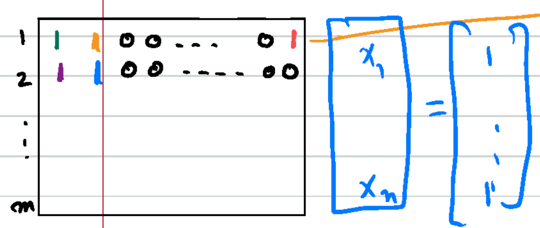
Solution: $x \in \{0,1\}^n$ s.t. $Ax = \mathbb{1}$.

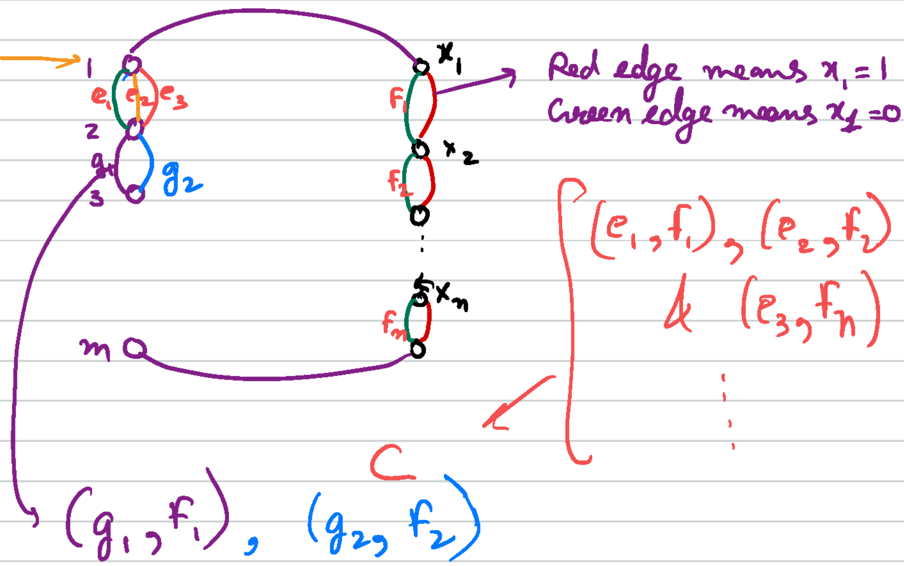
RC w PE

Instance: $G=(V,E)$ $\quad C \subseteq E \times E$

Solution: RC S in G s.t. $\forall (u,v) \in C$ exactly one of u, v is in S.

$A x = 1$





$1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_n = 1$

RC w PE $\longrightarrow$ RC

RC w PE

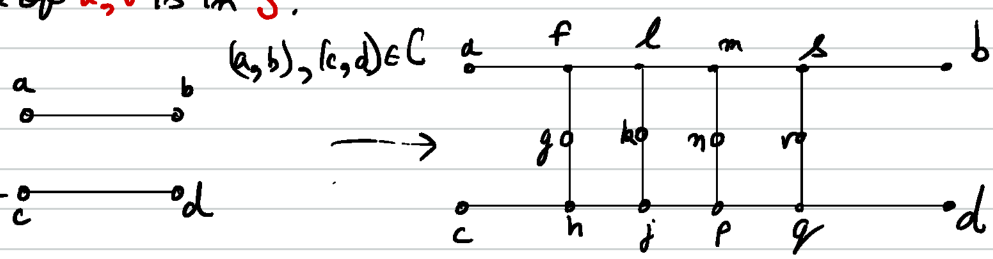
Instance: $G=(V,E)$ $\quad C \subseteq E \times E$

Solution: RC S in G s.t. $\forall (u,v) \in C$ exactly one of u, v is in S.

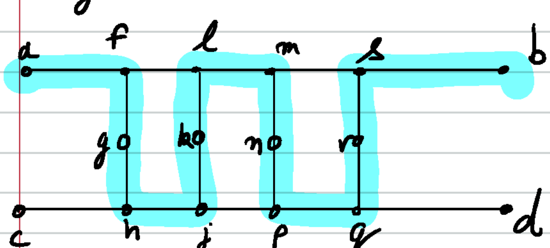
RC

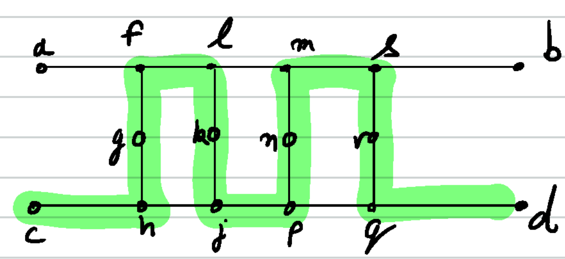
Instance: $G=(V,E)$

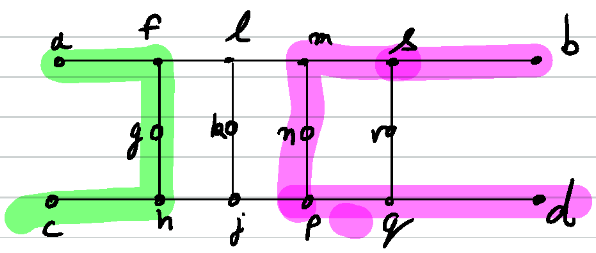
Solution: Cycle S visiting each vertex exactly once.

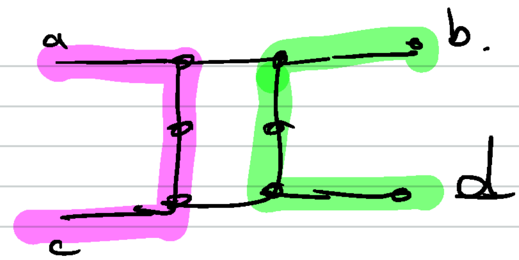


two paired edges









RC $\longrightarrow$ Travelling Salesman Problem (TSP)

Instance : $G=(V,E)$

Solution: Cycle S visiting each vertex exactly once.

Instance: distances d_{ij} & a bound B

Solution: a permutation $\tau:\{1\ldots n\}\to\{1..n\}$ s.t.

$$d_{\tau(1)\tau(2)} + d_{\tau(2)\tau(3)} \cdots d_{\tau(n)\tau(1)} \le B$$

$G \longrightarrow$

$d_{ij} = 1$ if $(i,j)\in E$

$d_{ij} = 1+\alpha$ if $(i,j)\notin E$

$B = |V|$

ZOE $\longrightarrow$ Subset Sum

Instance: $A \in \{0,1\}^{m \times n}$

Solution: $x \in \{0,1\}^n$ s.t. $A x = \mathbb{1}$.

Instance: $a_1, \dots, a_n$, w

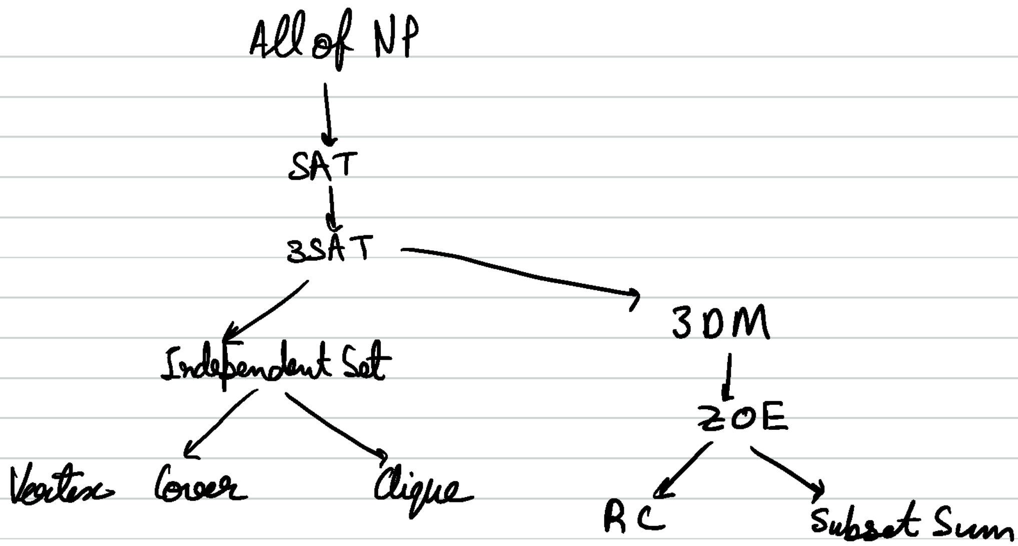
Solution: $x_1, \dots, x_n \in \{0,1\}$ s.t. $\sum a_i x_i = w$.

$$A = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & & \\ 1 & & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$a_i = \sum A_{ij} 2^j \qquad w = \sum 2^j$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ \begin{array}{cccc} 18 & 5 & 4 & 8 \end{array} \qquad\qquad\qquad 31 \end{array}$$

$$a_i = \sum A_{ij} (n+1)^j \qquad w = \sum (n+1)^j$$


All of NP
SAT
3SAT
3DM
Independent Set
ZOE
Vertex Cover
Clique
RC
Subset Sum