CS 170 DIS 1

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1 (★★★) Asymptotic notation

(a) For each pair of functions $f(n)$ and $g(n)$, state whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. For example, for $f(n) = n^2$ and $g(n) = 2n^2 - n + 3$, write $f(n) = \Theta(g(n))$.

- $f(n) = n$ and $g(n) = n^2 - n$
- $f(n) = n^2$ and $g(n) = n^2 + n$
- $f(n) = 8n$ and $g(n) = n \log n$
- $f(n) = 2^n$ and $g(n) = n^2$
- $f(n) = 3^n$ and $g(n) = 2^{2n}$

(b) For each of the following, state the order of growth using Θ notation, e.g. $f(n) = \Theta(n)$.

- $f(n) = 50$
- $f(n) = n^2 - 2n + 3$
- $f(n) = n + \cdots + 3 + 2 + 1$
- $f(n) = n^{100} + 1.01^n$
- $f(n) = n^{1.1} + n \log n$
- $f(n) = (g(n))^2$ where $g(n) = \sqrt{n} + 5$

Solution:

(a)
- $f(n) = O(g(n))$
- $f(n) = \Theta(g(n))$
- $f(n) = O(g(n))$
- $f(n) = \Omega(g(n))$
- $f(n) = O(g(n))$

(b)
- $f(n) = \Theta(1)$
- $f(n) = \Theta(n^2)$
- $f(n) = \frac{(n+1)n}{2} = \Theta(n^2)$
- $f(n) = \Theta(1.01^n)$
- $f(n) = \Theta(n^{1.1})$
- $f(n) = n + 10\sqrt{(n)} + 25 = \Theta(n)$

2 Asymptotic Bound Practice

Prove that for any $\epsilon > 0$ we have $\log x = O(x^\epsilon)$.

Solution:

Observe that $x > \log x \forall x > 0$. We can see this by taking finding the minimum of the function $x - \log x$ over the range $(0, \inf)$ using some calculus (find the critical points, then check concavity). The minimizing x is 1, with value 1.

If $x > \log x$, then we have that $\log x^\epsilon < x^\epsilon$, and therefore $\epsilon \log x < x^\epsilon$. It follows that a constant factor times x^ϵ is always larger than $\log x$ for $x > 0$. This proves $\log x = O(x^\epsilon)$.

Here is an alternate argument, using l'Hopital's rule:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\log x}{x^\epsilon} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\log x}{\frac{d}{dx}x^\epsilon} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{\epsilon x^{\epsilon-1}} \\
&= \lim_{x\to\infty} \frac{1}{\epsilon x^\epsilon} = 0
\end{aligned}$$

And so therefore $\log x = O(x^\epsilon)$.

3 Bounding Sums

Let $f(\cdot)$ be a function. Consider the equality

$$\sum_{i=1}^{n} f(i) \;=\; \Theta(f(n)),$$

Give a function f_1 such that the equality holds, and a function f_2 such that the equality does not hold.

Solution: There are many possible solutions.

$f_1(i) = 2^i$: $\sum_{i=1}^n 2^i = 2^{n+1} - 2 = \Theta(2^n)$.

$f_2(i) = i$: $\sum_{i=1}^n i = \frac{n(n+1)}{2} = \Theta(n^2) \neq \Theta(n)$.