Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Asymptotics and Limits

If we would like to prove asymptotic relations instead of just using them, we can use limits.

Asymptotic Limit Rules: If $f(n), g(n) \geq 0$:

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = \mathcal{O}(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, for some $c > 0$, then $f(n) = \Theta(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) = \Omega(g(n))$.

Note that these are all sufficient (and not necessary) conditions involving limits, and are not true definitions of $\mathcal{O}$, Θ, and Ω. We highly recommend checking on your own that these statements are correct!)

(a) Prove that $n^3 = \mathcal{O}(n^4)$.

Solution:

$$\lim_{n\to\infty} \frac{n^3}{n^4} = \lim_{n\to\infty} \frac{1}{n} = 0$$

So $f(n) = \mathcal{O}(g(n))$

(b) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$.

Solution: Let $f(n) = 3n$ and $g(n) = 5n$. Then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{3}{5}$, meaning that $f(n) = \Theta(g(n))$. However, it's still true in this case that $f(n) = \mathcal{O}(g(n))$ (just by the definition of Θ).

(c) Prove that for any $c > 0$, we have $\log n = \mathcal{O}(n^c)$.

Hint: Use L'Hôpital's rule: If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ (if the RHS exists)

Solution: By L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\log n}{n^c} = \lim_{n\to\infty} \frac{n^{-1}}{cn^{c-1}} = \lim_{n\to\infty} \frac{1}{cn^c} = 0$$

Therefore, $\log n = \mathcal{O}(n^c)$.

(d) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist. In this case, you would be unable to use limits to prove $f(n) = \mathcal{O}(g(n))$.

Hint: think about oscillating functions!

Solution: Let $f(x) = x(\sin x + 1)$ and $g(x) = x$. As $\sin x + 1 \leq 2$, we have that $f(x) \leq 2 \cdot g(x)$ for $x \geq 0$, so $f(x) = \mathcal{O}(g(x))$.

However, if we attempt to evaluate the limit, $\lim_{x\to\infty} \frac{x(\sin x+1)}{x} = \lim_{x\to\infty} \sin x + 1$, which does not exist (sin oscillates forever).

2 Asymptotic Complexity Comparisons

(a) Order the following functions so that for all i, j, if f_i comes before f_j in the order then $f_i = O(f_j)$. Do not justify your answers.

- $f_1(n) = 3^n$
- $f_2(n) = n^{\frac{1}{3}}$
- $f_3(n) = 12$
- $f_4(n) = 2^{\log_2 n}$
- $f_5(n) = \sqrt{n}$
- $f_6(n) = 2^n$
- $f_7(n) = \log_2 n$
- $f_8(n) = 2^{\sqrt{n}}$
- $f_9(n) = n^3$

As an answer you may just write the functions as a list, e.g. $f_8, f_9, f_1, \ldots$

Solution: $f_3, f_7, f_2, f_5, f_4, f_9, f_8, f_6, f_1$

(b) In each of the following, indicate whether $f = O(g)$, $f = \Omega(g)$, or both (in which case $f = \Theta(g)$). ***Briefly*** justify each of your answers. Recall that in terms of asymptotic growth rate, constant $<$ logarithmic $<$ polynomial $<$ exponential.

	$f(n)$	$g(n)$
(i)	$\log_3 n$	$\log_4(n)$
(ii)	$n\log(n^4)$	$n^2\log(n^3)$
(iii)	$\sqrt{n}$	$(\log n)^3$
(iv)	$n + \log n$	$n + (\log n)^2$

Solution:

(i) $f = \Theta(g)$; using the log change of base formula, $\frac{\log n}{\log 3}$ and $\frac{\log n}{\log 4}$ differ only by a constant factor.

(ii) $f = O(g)$; $f(n) = 4n\log(n)$ and $g(n) = 3n^2\log(n)$, and the polynomial in g has the higher degree.

(iii) $f = \Omega(g)$; any polynomial dominates a product of logs. We can also obtain this result via the limit proof below:

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{\sqrt{n}}{(\log n)^3} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{3(\log n)^2 \cdot \frac{1}{n}} && \text{[L'Hôpital's rule]} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{6(\log n)^2} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{24\log n} && \text{[L'Hôpital's rule again]} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{48} && \text{[L'Hôpital's rule one more time]} \\
&= \infty
\end{aligned}$$

(iv) $f = \Theta(g)$; Both f and g grow as $\Theta(n)$ because the linear term dominates the other. We can also obtain this result via the limit proof below:

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n+\log n}{n+(\log n)^2} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2\log n}{n}} && \text{[L'Hôpital's rule]} \\
&= 1
\end{aligned}$$

3 Recurrence Relations

Solve the following recurrence relations, assuming base cases $T(0) = T(1) = 1$:

(a) $T(n) = 2 \cdot T(n/2) + O(n)$

Solution: We can use the Master Theorem here! Noting that $a = 2, b = 2$, and $d = 1$, we have that $\log_b a = \log_2(2) = 1 = d$. Thus, via the Master Theorem, we have

$$T(n) = O(n^d \log n) = O(n \log n)$$

(b) $T(n) = T(n-1) + n$

Solution: Since we can't use Master Theorem here, we use the "unravelling" strategy as follows:

$$\begin{aligned}
T(n) &= T(n-1) + n \\
&= (T(n-2) + (n-1)) + n \\
&= ((T(n-3) + (n-2)) + (n-1)) + n \\
&= \cdots \text{Unravelling} \cdots \\
&= T(1) + 2 + 3 + \cdots (n-2) + (n-1) + n \\
&= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
&= \sum_{i=1}^{n} i \\
&= \frac{n(n+1)}{2} \\
&= O(n^2)
\end{aligned}$$

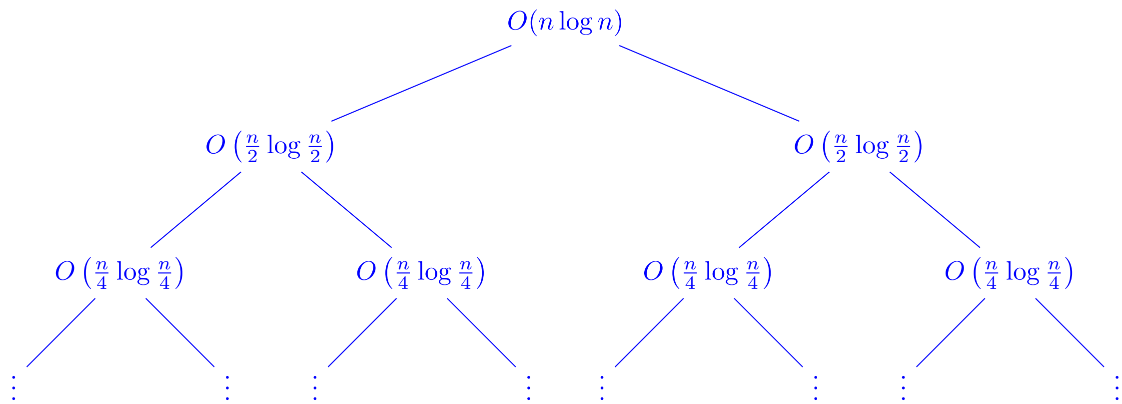
(c) $T(n) = 3 \cdot T(n-2) + 5$

Solution: More unravelling! Here we go again:

$$\begin{aligned}
T(n) &= 3T(n-2) + 5 \\
&= 3^2 T(n-4) + 5 \cdot 3 + 5 \\
&= 3^3 T(n-6) + 5 \cdot 3^2 + 5 \cdot 3 + 5 \\
&= 3^4 T(n-8) + 5 \cdot 3^3 + 5 \cdot 3^2 + 5 \cdot 3 + 5 \\
&= \cdots \\
&= 3^{\lfloor n/2 \rfloor} T(n \bmod 2) + 5 \cdot 3^{\lfloor n/2 \rfloor - 1} + 5 \cdot 3^{\lfloor n/2 \rfloor - 2} + \cdots + 5 \cdot 3^2 + 5 \cdot 3 + 5 \\
&= 1 \cdot 3^{\lfloor n/2 \rfloor} + 5 \cdot 3^{\lfloor n/2 \rfloor - 1} + 5 \cdot 3^{\lfloor n/2 \rfloor - 2} + \cdots + 5 \cdot 3^2 + 5 \cdot 3 + 5 \\
&= 3^{\lfloor n/2 \rfloor} + \frac{5 \cdot (3^{\lfloor n/2 \rfloor} - 1)}{3 - 1} \\
&= \frac{7}{2} \cdot 3^{\lfloor n/2 \rfloor} - \frac{5}{2} \\
&= O(3^{n/2})
\end{aligned}$$

(d) $T(n) = 2 \cdot T(n/2) + O(n \log n)$

Solution: We use the tree drawing technique here. We draw the following recursion tree, where the nodes represent the work done by a recursive call:



Summing up all the levels, we get

$$\begin{aligned}
T(n) &= O(n\log n) + O\left(n\log\left(\frac{n}{2}\right)\right) + O\left(n\log\left(\frac{n}{4}\right)\right) + \cdots + O(1) \\
&= O\left(\sum_{i=0}^{\lfloor \log n \rfloor} n\log\left(\frac{n}{2^i}\right)\right) \\
&= O\left(\sum_{i=0}^{\lfloor \log n \rfloor} n\left(\log n - i\right)\right) \\
&= O\left(n\left(\log^2 n - \sum_{i=0}^{\lfloor \log n \rfloor} i\right)\right) \\
&= O\left(n\left(\log^2 n - \frac{1}{2}\log^2 n\right)\right) \\
&= O(n\log^2 n)
\end{aligned}$$

(e) $T(n) = 3T(n^{1/3}) + O(\log n)$

Solution: Since we're recursing on a weird $n^{1/3}$ cubic root, we should use a *change of variables* to get mold the recurrence into something that's more manageable. Let's try the substitution $x = \log n$, so that

$$n^{1/3} = (e^x)^{1/3} = e^{x/3}.$$

Then, our recurrence becomes $T(e^x) = 3T(e^{x/3}) + O(x)$. Now, let us define $S(x) = T(e^x) = T(n)$, which has the following (nice) recurrence:

$$S(x) = 3S(x/3) + O(x).$$

Look at this; we can apply the Master Theorem! Solving, we get $S(x) = O(x \log x)$. Finally, note that we want to find $T(n)$ and not $S(x)$, so we plug $x = \log n$ back in as follows:

$$T(n) = S(x) = O(x \log x) = O(\log n \cdot \log \log n).$$

(f) $T(n) = T(n-1) + T(n-2)$

Solution: We apply the squeeze + guess & check method. First, we can lower bound it by, $T(n) \geq 2T(n-2)$, so we know $T(n) = \Omega(2^{n/2})$. We can also upper-bound it by $T(n) \leq 2T(n-1)$, which gives $T(n) = O(2^n)$.

Hence, $T(n) = 2^{\Theta(n)}$. However, we can actually compute a more precise runtime! Since we know the runtime is exponential with respect to n, we can write the runtime in the form $T(n) = \Theta(a^n)$. Then, plugging this into the recurrence, we have

$$a^n = a^{n-1} + a^{n-2}$$

$$a^2 = a + 1$$

$$a^2 - a - 1 = 0$$

where we can divide by a^{n-2} since $a \neq 0$. By the quadratic formula, we get that $a = \frac{1 \pm \sqrt{5}}{2}$. Since a must be positive, we conclude that $a = \frac{1+\sqrt{5}}{2}$ and thus

$$T(n) = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$$

