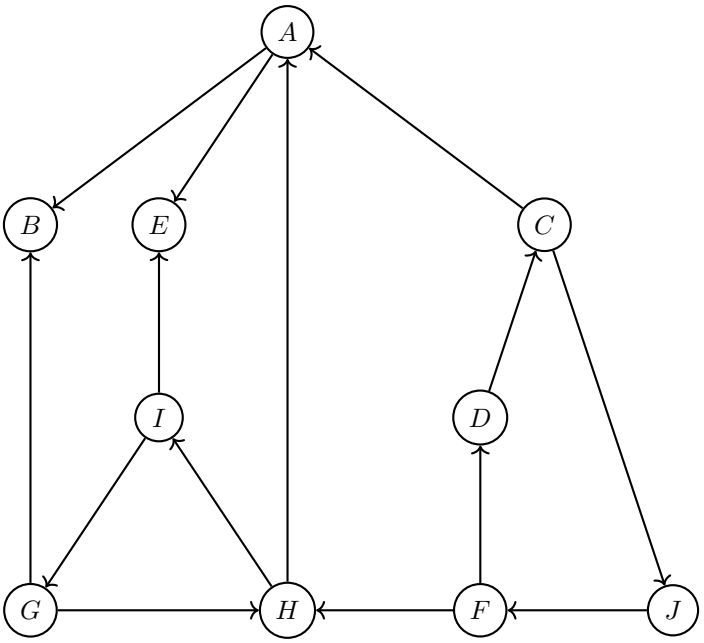
Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Graph Traversal



(a) Recall that given a DFS tree, we can classify edges into one of four types:

- Tree edges are edges in the DFS tree,
- Back edges are edges (u, v) not in the DFS tree where v is the ancestor of u in the DFS tree
- Forward edges are edges (u, v) not in the DFS tree where u is the ancestor of v in the DFS tree
- Cross edges are edges (u, v) not in the DFS tree where u is not the ancestor of v, nor is v the ancestor of u.

For the directed graph above, perform DFS starting from vertex A, breaking ties alphabetically. As you go, label each node with its pre- and post-number, and mark each edge as **T**ree, **B**ack, **F**orward or **C**ross.

(b) A strongly connected component (SCC) is defined as a subset of vertices in which there exists a path from each vertex to another vertex. What are the SCCs of the above graph?

(c) Collapse each SCC you found in part (b) into a meta-node, so that you end up with a graph of the SCC meta-nodes. Draw this graph below, and describe its structure.

2 Graph Short Answer

For each of the following, either prove the statement is true or give a counterexample to show it is false. Note that pre(v) and post(v) denote that pre-order and post-order values of v.

(a) If (u, v) is an edge in an undirected graph and during DFS, post(v) < post(u), then u is an ancestor of v in the DFS tree.

(b) In a directed graph, if there is a path from u to v and pre(u) < pre(v) then u is an ancestor of v in the DFS tree.

(c) We can modify the SCC algorithm from lecture, so that, the DFS on G is done in decreasing pre-order of G instead of decreasing post-order of G^R. This modified algorithm is also correct.

(d) We can modify the SCC algorithm from lecture so that the first DFS is run on G and the second DFS is run on G^R. This modified algorithm is also correct.

3 Finding Clusters

We are given a directed graph $G = (V, E)$, where $V = \{1, \ldots, n\}$, i.e. the vertices are integers in the range 1 to n. For every vertex i we would like to compute the value $m(i)$ defined as follows: $m(i)$ is the smallest j such from which you can reach vertex i. (As a convention, we assume that i is reachable from i.)

(a) Show that the values $m(1), \ldots, m(n)$ can be computed in $O(|V| + |E|)$ time. **Please provide an algorithm description and runtime analysis; proof of correctness is not required.**

(b) Suppose we instead define $m(i)$ to be the smallest j that can be reached from i, instead of the smallest j from which you can reach i. How should you modify your answer to part (a) to work in this case?

4 Not So Exciting Scheduling

PNP University requires students to finish all prerequisites for a certain class before taking it; however, they made some mistakes when assigning prerequisites. Thus, some classes at PNP University are potentially *NP (Not Possible to take)* because it may be impossible to take all its prerequisite classes whilst following proper prerequisite rules. For example, if CS 61A is a prereq of CS 61B, CS 61B is a prereq of CS 170, and CS 170 is mistakenly listed as a prereq of CS 61A, then all three classes would be NP.

Due to these mistakes, students wish to figure out whether their classes can all be taken or not while following prerequisites. Their n classes are labelled with unique identifiers $\{c_1, c_2, \ldots, c_n\}$, and the set of m prerequisites in the form $[c_i, c_j]$ indicate that c_i must be taken before c_j.

Design an algorithm that outputs a potential scheduling of classes (i.e. an ordering of classes to take) if there are no NP classes; return false otherwise. **Please provide an algorithm description and a runtime analysis; proof of correctness is not required.**

5 Biconnected Components

Consider any undirected connected graph $G = (V, E)$. We say that an edge $(u, v) \in E$ is *critical* if removing it disconnects the graph. In other words, the graph $(V, E \setminus (u, v))$ is no longer connected. Similarly, we call a vertex $v \in V$ critical if removing v (and all its incident edges) leaves the graph disconnected.

(a) Can you always find a vertex $v \in V$ that is **not** critical? What about an edge that is not critical?

(b) Give a linear time algorithm to find all the critical edges of G.

(c) Modify your algorithm above to find all the critical vertices of G.

(d) A biconnected component of G is a connected subgraph that does not contain critical vertices; in other words, if you remove a vertex from the component, then it will remain connected. If we collapse all the biconnected components of G into a meta-node (similar to what we did with SCCs), what does the resulting graph look like?