

Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Planting Trees

This problem will guide you through the process of writing a dynamic programming algorithm.

You have a garden and want to plant some apple trees in your garden, so that they produce as many apples as possible. There are n adjacent spots numbered 1 to n in your garden where you can place a tree. Based on the quality of the soil in each spot, you know that if you plant a tree in the i th spot, it will produce exactly x_i apples. However, each tree needs space to grow, so if you place a tree in the i th spot, you can't place a tree in spots $i - 1$ or $i + 1$. What is the maximum number of apples you can produce in your garden?

(a) Give an example of an input for which:

- Starting from either the first or second spot and then picking every other spot (e.g. either planting the trees in spots 1, 3, 5... or in spots 2, 4, 6...) does not produce an optimal solution.
- The following algorithm does not produce an optimal solution: While it is possible to plant another tree, plant a tree in the spot where we are allowed to plant a tree with the largest x_i value.

(b) To solve this problem, we'll think about solving the following, more general problem: "What is the maximum number of apples that can be produced using only spots 1 to i ?" Let $f(i)$ denote the answer to this question for any i . Define $f(0) = 0$, as when we have no spots, we can't plant any trees. What is $f(1)$? What is $f(2)$?

(c) Suppose you know that the best way to plant trees using only spots 1 to i does not place a tree in spot i . In this case, express $f(i)$ in terms of x_i and $f(j)$ for $j < i$. (Hint: What spots are we left with? What is the best way to plant trees in these spots?)

(d) Suppose you know that the best way to plant trees using only spots 1 to i places a tree in spot i . In this case, express $f(i)$ in terms of x_i and $f(j)$ for $j < i$.

- (a) Alice decides to use a greedy strategy: “on my turn, I will take whichever of the first and last card is worth more points”. Show a small counterexample ($n \leq 5$) where Alice will lose if she plays this greedy strategy, assuming Alice goes first and Bob plays optimally, but she could have won if she had played optimally.

- (b) After seeing your counterexample, Alice decides to use dynamic programming to play optimally, assuming Bob also plays optimally. Write a dynamic program that computes how many points Alice ends up with, and state its runtime.

(Hint: Let $s(i, j)$ denote Alice’s score at the end of the game if the game consists only of cards i to j and Alice goes first. To write a recurrence for $s(i, j)$, first think about what Bob should do on his turn. Then use that to decide what Alice should do on her turn).

4 Counting Targets

We call a sequence of n integers x_1, \dots, x_n *valid* if each x_i is in $\{1, \dots, m\}$.

- (a) Give a dynamic programming-based algorithm that takes in n, m and “target” T as input and outputs the number of distinct valid sequences such that $x_1 + \dots + x_n = T$. Your algorithm should run in time $O(m^2 n^2)$. Note that you can assume $T \leq mn$ since no valid sequences sum to more than mn .

- (b) **Challenge:** Give an algorithm for the problem in part (a) that runs in time $O(mn^2)$.