

*Note:* Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

## 1 Linear Programming Basics

Plot the feasible region and identify the optimal solution for the following linear program.

$$\begin{aligned} & \text{maximize } 5x + 3y \\ \text{s.t. } & 5x - 2y \geq 0, \quad x + y \leq 7, \quad x \leq 5, \quad x \geq 0, \quad y \geq 0 \end{aligned}$$

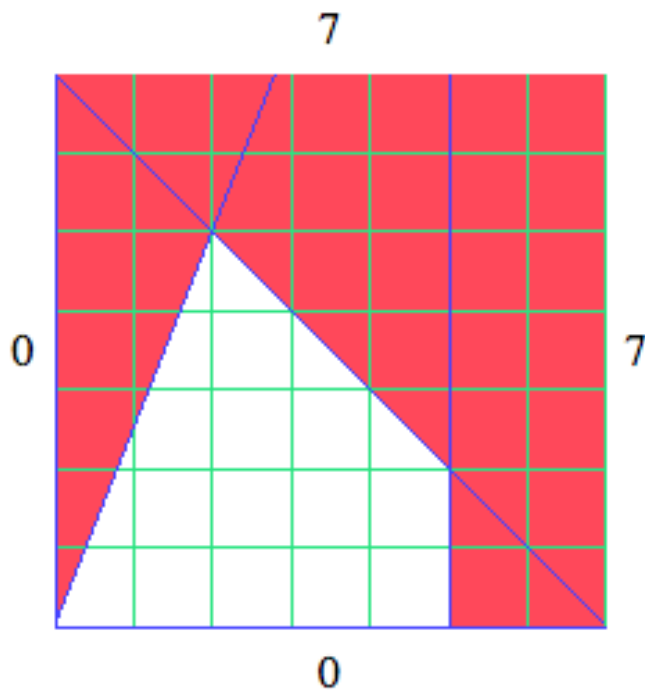
Suppose we want to maximize

$$\min\{5x, 3y\}$$

instead, subject to the same constraints. Describe how we can modify the LP to solve this problem by changing the objective, adding one new variable, and adding two new constraints.

**Solution:**

Here is the feasible region:



The feasible region is shown in white.

We know our optimum must occur at a vertex. The vertex of (5, 2) with objective value 31 achieves the maximum value.

## 2 Standard Form LP

Recall that any Linear Program can be reduced to a more constrained *standard form* where all variables are nonnegative, the constraints are given by equations and the objective is that of minimizing a cost function.

More formally, our variables are  $x_i$ . Our objective is  $\min c^\top x = \sum_i c_i x_i$  for some constants  $c_i$ . The  $j$ th constraint is  $\sum_i a_{ij} x_i = b_j$  for some constants  $a_{ij}, b_j$ . Finally, we also have the constraints  $x_i \geq 0$ .

An example standard form LP:

$$\begin{aligned} & \text{minimize } 5x_1 + 3x_2 \\ \text{s.t. } & x_1 + x_2 - x_3 = 1, \quad -x_1 + 2x_2 + x_4 = 0, \quad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

For each of the subparts, what system of variables, constraints, and objectives would be equivalent to the following:

- (a) Max Objective:  $\max \sum_i c_i x_i$
- (b) Min Max Objective:  $\min \max(y_1, y_2)$
- (c) Upper Bound on Variable:  $x_1 \leq b_1$
- (d) Lower Bound on Variable:  $x_2 \geq b_2$
- (e) Bounded Variable:  $b_2 \leq x_3 \leq b_1$
- (f) Inequality Constraint:  $x_1 + x_2 + x_3 \leq b_3$
- (g) Unbounded Variable:  $x_4 \in R$

**Solution:**

- (a)  $\min -\sum_i c_i x_i$
- (b)  $\min t, \quad x \leq t, \quad y \leq t$
- (c)  $x_1 + s_1 = b_1, \quad s_1 \geq 0$
- (d)  $-x_2 + s_2 = -b_2, \quad s_2 \geq 0$
- (e) Break it into two inequalities  $x_3 \leq b_1$  and  $x_3 \geq b_2$  and use the parts above
- (f)  $x_1 + x_2 + x_3 + s_1 = b_3, \quad s_1 \geq 0$
- (g) Replace  $x_4$  by  $x^+ - x^-$  along with  $x^+ \geq 0, \quad x^- \geq 0$

## 3 An LP for Minimum Spanning Tree

Consider the minimum spanning tree problem, where we are given an undirected graph  $G$  with edge weights  $w_{u,v}$  for every pair of vertices  $u, v$ .

An *integer* linear program that solves the minimum spanning tree problem is as follows:

$$\begin{aligned} & \text{Minimize } \sum_{(u,v) \in E} w_{u,v} x_{u,v} \\ & \text{subject to } \sum_{\{u,v\} \in E: u \in S, v \in V \setminus S} x_{u,v} \geq 1 \quad \text{for all } S \subseteq V \text{ with } 0 < |S| < |V| \\ & x_{u,v} \in \{0, 1\}, \quad \forall (u, v) \in E \end{aligned}$$

- (a) Show how to obtain a minimum spanning tree  $T$  of  $G$  from an optimal solution of the ILP, and prove that  $T$  is indeed an MST. Why do we need the constraint  $x_{u,v} \in \{0, 1\}$ ?
- (b) How many constraints does the program have?
- (c) Suppose that we *replaced* the binary constraint on each of the decision variables  $x_{u,v}$  with the pair of constraints:

$$0 \leq x_{u,v} \leq 1, \quad \forall (u,v) \in E$$

How does this affect the optimal value of the program? Give an example of a graph where the optimal value of the relaxed linear program differs from the optimal value of the integer linear program.

**Solution:**

- (a)  $T = \{(u,v) \in E : x_{u,v} = 1\}$ . The first constraint ensures that  $T$  is connected (there is at least one edge crossing every cut). Furthermore, note that any optimal solution will correspond to a tree, because any solution containing a cycle can be improved. Moreover, every spanning tree  $T$  is a feasible solution of the ILP. The objective is the weight of  $T$ , and so the optimum is the MST. We need  $x_{u,v} \in \{0, 1\}$  because it's not clear what you'd do with a fractional edge.
- (b) There are  $2^{|V|} + |E| - 1 = \Theta(2^{|V|})$  constraints.
- (c)  $v_{LP} \leq v_{ILP}$ . The new linear program solution's objective value  $v_{LP}$  is at most the integer linear program's objective value  $v_{ILP}$ , because every feasible solution of the ILP is a feasible solution of the LP.

One example is a cycle with 3 nodes,  $w_{u,v} = 1, \forall u,v \in E$ . The optimal ILP formulation picks any two of the edges for a total objective cost of 2. The optimal LP formulation picks  $x_{u,v} = \frac{1}{2}$  for all edges, for a total objective cost of  $\frac{3}{2}$ .

## 4 Vertex Cover Rounding

In the vertex cover problem, we are given a graph  $G$ , and our goal is to find the smallest set of vertices  $S$  such that every edge has at least one endpoint in  $S$ .

- (a) Let's write an integer linear program (ILP) for the vertex cover problem. There will be one variable  $x_v$  for every vertex, and we will set  $x_v = 1$  if  $v$  is in our solution and  $x_v = 0$  if  $v$  is not in our solution.

To finish writing the ILP, what is the objective function? What are the constraints?

- (b) If we replace the requirement  $x_v \in \{0, 1\}$  with the relaxed requirement  $x_v \in [0, 1]$ , we get a normal linear program (LP). LPs can be solved efficiently but ILPs cannot.

However, this efficiency does not come for free: Give an example of a graph where the optimal solution to this LP has objective function smaller than the size of the minimum vertex cover. (This suggests that we can't solve the vertex cover problem by just solving an LP for it.)

- (c) Suppose someone solves this LP (*not* the ILP) and hands you the solution. Given *only* the solution and not  $G$ , how can we compute a vertex cover whose size is at most twice the fractional solution's objective function? e.g. if you're handed a solution which gets an objective function of 7, you should output a vertex cover of size at most 14.

(Hint: Include vertices whose  $x_v$  are large enough).

**Solution:**

- (a) The objective is  $\min \sum_{v \in V} x_v$ , i.e. minimize the number of vertices used. The constraint is for any edge  $e = (u, v)$ ,  $x_u + x_v \geq 1$ , i.e. for any edge we must include at least one of its endpoints.
- (b) In a complete graph, we can set all  $x_v = 1/2$  and get an LP with objective function  $n/2$ , but any vertex cover is size at least  $n - 1$ .
- (c) If  $x_v \geq 1/2$ , we include  $v$  in our vertex cover, and otherwise don't include it.

This is a feasible vertex cover since for any edge  $(u, v)$ ,  $x_u + x_v \geq 1$ , which means at least one of  $x_u, x_v$  is at least  $1/2$  in any feasible LP solution, so one of  $(u, v)$ 's endpoints gets included in our vertex cover.

On the other hand, for every vertex in our solution, it contributes 1 to the size of our vertex cover and at least  $1/2$  to the fractional solution's objective function, so the vertex cover's size is at most twice the fractional solution's objective function.