Task Scheduling

Downtown Berkeley can be represented as a coordinate plane where each coordinate \((x, y)\) is integer and bounded by \(1 \leq x \leq r\) and \(1 \leq y \leq r\). In order to travel from \((x, y)\) to \((x', y')\), it takes \(|x - x'| + |y - y'|\) minutes. You live at point \((x_0, y_0)\), and there are \(n\) tasks labelled \(1, 2, \ldots, n\) scattered around Berkeley at points \((x_i, y_i)\) that must be completed at exactly integer time \(t_i\), where \(t_i < t_{i+1}\) for all \(i\). You can finish a task immediately if you are at that location. You would like to figure out the maximal amount of tasks you can finish.

(a) Give a three-part \(O(n^2)\) time dynamic programming algorithm to solve this problem.

(b) Improve this algorithm to \(O(nr)\) time. Just a description and brief explanation of why it works is sufficient. \textit{Hint: the city size is limited and tasks happen at distinct times.}
2 LP Basics

Linear Program. A linear program is an optimization problem that seeks the optimal assignment for a linear objective over linear constraints. Let $x \in \mathbb{R}^n$ be the set of variables and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. The canonical form of a linear program is

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]

Any linear program can be written in canonical form.

Let’s check this is the case:

(i) What if the objective is maximization?
(ii) What if you have a constraint $Ax \leq b$?
(iii) What about $Ax = b$?
(iv) What if the constraint is $x \leq 0$?
(v) What about unconstrained variables $x \in \mathbb{R}$?

Dual. The dual of the canonical LP is

\[
\begin{align*}
\text{maximize} & \quad b^\top y \\
\text{subject to} & \quad A^\top y \leq c \\
& \quad y \geq 0
\end{align*}
\]

Weak duality: The objective value of any feasible dual $\leq$ objective value of any feasible primal

Strong duality: The optimal objective values of these two are equal.

Both are solvable in polynomial time by the Ellipsoid or Interior Point Method.
3 Taking a Dual

Consider the following linear program:

\[
\begin{align*}
\text{max} & \quad 4x_1 + 7x_2 \\
& x_1 + 2x_2 \leq 10 \\
& 3x_1 + x_2 \leq 14 \\
& 2x_1 + 3x_2 \leq 11 \\
& x_1, x_2 \geq 0
\end{align*}
\]

Construct the dual of the above linear program.

4 Huffman and LP

Consider the following Huffman code for characters \(a, b, c, d\): \(a = 0, b = 10, c = 110, d = 111\).

Let \(f_a, f_b, f_c, f_d\) denote the fraction of characters in a file (only containing these characters) that are \(a, b, c, d\) respectively. Write a linear program with variables \(f_a, f_b, f_c, f_d\) to solve the following problem: What values of \(f_a, f_b, f_c, f_d\) that can generate this Huffman code result in the Huffman code using the most bits per character?
5 Job Assignment

There are $I$ people available to work $J$ jobs. The value of person $i$ working 1 day at job $j$ is $a_{ij}$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Each job is completed after the sum of the time of all workers spend on it add up to be 1 day, though partial completion still has value (i.e. person $i$ working $c$ portion of a day on job $j$ is worth $a_{ij}c$). The problem is to find an optimal assignment of jobs for each person for one day such that the total value created by everyone working is optimized. No additional value comes from working on a job after it has been completed.

(a) What variables should we optimize over? I.e. in the canonical linear programming definition, what is $x$?

(b) What are the constraints we need to consider? Hint: there are three major types.

(c) What is the maximization function we are seeking?